BUSINESS MATHS & STATISTICS (TC3)

January 2014

TECHNICIAN DIPLOMA IN ACCOUNTING

INSTITUTE OF CHARTERED ACCOUNTANTS IN MALAWI (ICAM)

NOT FOR SALE
PREFACE

INTRODUCTION

The Institute noted a number of difficulties faced by students when preparing for the Institute’s examinations. One of the difficulties has been the unavailability of study manuals specifically written for the Institute’s examinations. In the past students have relied on text books which were not tailor-made for the Institute’s examinations and the Malawian environment.

AIM OF THE MANUALS

The manual has been developed in order to provide resources that will help the Institute’s students attain the needed skills. It is therefore recommended that each student should have their own copy.

HOW TO USE THE MANUAL

Students are being advised to read chapter by chapter since subsequent work often builds on topics covered earlier.

Students should also attempt questions at the end of the chapter to test their understanding. The manual will also be supported with a number of resources which students should keep checking on the ICAM website.
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AIM OF THE COURSE
To enable the student to understand mathematical and statistical principles and their applications in business.

OBJECTIVES
On completion of this module, the candidate will be able to:

- Solve business equations.
- Sketch graphs of business functions
- Solve business problems using techniques of sequences and series
- Use inequalities, where appropriate, to solve simple commercial situations
- Apply the concept of matrices in business
- Describe data collection techniques and sources of data
- Present data graphically and use data summarization techniques.
- Calculate measures of central tendency and dispersion
- Interpret measures of central tendency and dispersion
- Apply the concept of probability in solving business problems.
- Forecast using business data.
- Determine the degree of relationship between two variables.
- Interpret index numbers
- Perform investment appraisal.
- Apply calculus on revenue, cost and profit functions with the aim of finding optimum points.

FORMAT AND STANDARD OF THE EXAMINATION PAPER
The Business Mathematics and Statistics module will be assessed using a traditional 3 hour paper-based examination. The examination paper will consist of two sections; section A and section B. Section A will be compulsory and it will carry 60 marks. Section B will have 3 questions each carrying 20 marks. Candidates will be required to answer any 2 questions from section B.
SPECIFICATION GRID

This grid shows the relative weightings of topics within this course and should provide guidance regarding study time to be spent on each.

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Learning Outcomes

1. Functions, equations and graphs
   The candidate will be able to formulate and solve equations from real life situations.

   In the assessment, the candidate may be required to:
   i. formulate equations.
   ii. solve linear equations.
   iii. solve quadratic equations using graph, factorisation and formula.
   iv. solve exponential and logarithmic equations.
   v. solve compound interest problems using logarithms.

2. Sequences and series
   The candidate will be able to identify patterns in a given set of data and use the appropriate method to solve real-life problems.

   In the assessment, the candidate may be required to:
   i. Define an Arithmetic Progression
   ii. Identify an Arithmetic Progression
   iii. Identify a geometric progression
   iv. Determine the $n^{th}$ terms of arithmetic and geometric progressions
   v. Find the sum of a given AP
   vi. Find the sum of a given GP

3. Inequalities and Linear Programming
   The candidate will be able to use linear programming techniques to make managerial decisions.

   In the assessment, the candidate may be required to:
   i. formulate simple linear inequalities
   ii. solve linear inequalities
iii. apply inequalities to simple commercial situations  
iv. find the objective function for a given problem.  
v. come up with constraints from a given word problem.  
vi. plot inequalities and determine a feasible region for a given word problem.  
vii. Use graphical method to find the optimal solution to a linear programming problem.  

4. Matrices  
The candidate will be able to model relationships between financial or economic variables using a set of linear equations, represent them using matrices and solve such models.  

In the assessment, the candidate may be required to:  
i. Represent data with matrices  
ii. Add, subtract and multiply matrices  
iii. Apply the concept of matrices in manipulating commercial data  
iv. Find the determinant of matrices up to 3 by 3.  
v. Find the inverse of $2 \times 2$ and $3 \times 3$ matrices.  
vi. Solve systems of linear equations (up to 3 variables) using the inverse method and Cramer’s rule.  

5. Sampling and data collection  
The candidate will be able to choose among the various methods employed in choosing the subjects for an investigation and differentiate the different ways of collecting data.  

In the assessment, the candidate may be required to:  
i. Distinguish between data and information  
ii. Classify data  
iii. Distinguish between data collection methods and select a suitable method  
iv. State stages in statistical investigation  

6. Data Presentation  
The candidate will be able to present data using various data presentation techniques.  

In the assessment, the candidate may be required to:  
i. Present data using various techniques: pie chart, pictogram, bar chart, frequency tables, histogram, Ogive, Lorenz curve and Z chart.  
ii. Select an appropriate data presentation technique for specific data basing on type of data at hand, and advantages and disadvantages of the technique.  

7. Statistical Measures  
The candidate will be able to calculate and interpret measures of central tendency and dispersion.  

In the assessment, the candidate may be required to:  
i. Calculate measures of central tendency from simple data: mean, mode, median, geometric mean for grouped and un-grouped data.  
ii. Interpret measures of central tendency  
iii. Calculate measures of dispersion: range, mean deviation, variance and standard deviation, quartile deviation, coefficient of variation and skewness for grouped and un-grouped data.  
iv. Interpret measures of dispersions  
v. Compare distributions using summary measures
vi. Determine the skewness of a distribution: Pearson’s measure of skewness
vii. Interpret the skewness of a distribution

8. Probability
The candidate will be able to calculate and interpret various types of probability.

In the assessment, the candidate may be required to:
i. Describe the role of probability in decision making
ii. Describe the classical, empirical, and subjective approaches to probability
iii. Distinguish experiment, event and outcome
iv. Apply the rules of probability: addition and multiplication rules
v. Calculate marginal and conditional probabilities
vi. Use a tree diagram to organize and compute probabilities

9. Correlation and regression
The candidate will be able to determine the relationship between two numeric variables compute the strength of the relationship.

In the assessment, the candidate may be required to:
i. explain the meaning of regression analysis
ii. identify practical examples where regression analysis can be used
iii. plot scatter diagrams
iv. construct a simple linear regression model
v. prepare estimates of the unknown variable using the regression model
vi. compute and interpret the Pearson product moment correlation coefficient
vii. compute and interpret the Pearson product correlation coefficient
viii. compute and interpret the coefficient of determination

10. Times Series Analysis
The candidate will be able to use time series data to forecast events or activities.

In the assessment, the candidate may be required to:
i. Plot time series data
ii. Describe times series models
iii. Distinguish components of a time series: Trend, Seasonal, Random and cyclical variation.
iv. Decompose a time series into its components: Trend and Seasonal variations using moving averages.
v. Apply time series to make forecasts
vi. Compute and interpret de-seasonalised data.

11. Index Numbers
The candidate will be able to manipulate different published index series and construct new index series.

In the assessment, the candidate may be required to:
i. Explain what an index number is.
ii. Distinguish between base year and current year
iii. Construct single item indices (price and quantity).
iv. Differentiate between weighted and un-weighted indices.
v. Change the base of an index number
vi. Calculate the Laspeyres and Paasche Indices and explain the difference
vii. Measure changes in economic data using indices
viii. Adjust nominal money values into real terms (taking inflation into account)

12. Financial Mathematics
The candidate will be able to solve problems involving depreciation, interest calculations and investment appraisal.

In the assessment, the candidate may be required to:
i. Calculate interest, principal or period in given various combinations of parameters
ii. Describe the different techniques of depreciation.
iii. Depreciate an asset using the various depreciation techniques
iv. Appraise an investment using payback period, Net Present Values and Internal Rate of Return
v. Compare the various techniques of investment appraisal
vi. Calculate the maturity value of an annuity
vii. Calculate periodic payments for a sinking fund
viii. Calculate the fixed equal payment of annuity given the maturity value
ix. Describe amortisation as a method of debt repayment

13. Introduction to Calculus
The candidate will be able to apply the principles of differentiation and integration and apply these to rates of change of functions e.g profit function and interpret the results to determine when functions are at their minimum or maximum.

In the assessment, the candidate may be required to:
i. Differentiate functions up to the second derivative.
ii. Evaluate indefinite and definite integrals.
iii. Find minimum and maximum values of a given function.
iv. Apply calculus on revenue, cost and profit functions with the aim of finding optimum points.

REFERENCES
LEARNING OBJECTIVES

By the end of this chapter the student should be able to:

i. Use the common symbols used in calculations
ii. Use the rules governing arithmetic operations
iii. Solve problems involving powers

1.0 INTRODUCTION

Mathematics and statistics use a lot of symbols to represent values, concepts, processes and results management of which would be cumbersome if not impossible, if prose was used. Further calculations involved are governed by fundamental rules. This chapter introduces the common symbols and basic rules required for meaningful calculations.

1.1 MATHEMATICAL NOTATION

The term mathematical notation refers to a system of symbolic representations of mathematical objects and ideas. Mathematical notations are used in mathematics, the physical sciences, engineering, and economics and accounting.

In its broad sense the term mathematical notations include relatively simple symbolic representations, such as numbers 1 and 2, function symbols such as $\times$ and $\div$; conceptual symbols, such as $\frac{dy}{dx}$, equations and variables.

This chapter will introduce various symbols and explain a few functional notations.

1.2 MATHEMATICAL AND STATISTICAL SYMBOLS

The following table shows mathematical and statistical symbols commonly used in Accounting and commerce.
1.3 IMPORTANT CONCEPTS AND RULES IN ARITHMETIC OPERATIONS

1.3.1 Number Systems

Certain sets of numbers are so frequently referred to that they are given special symbolic names.

**Natural Numbers:** This is the set of counting numbers 1, 2, 3, ..., It is usually denoted as \( \mathbb{N} \). So \( \mathbb{N} = \{1, 2, 3, \ldots\} \). Elements of this set can be multiplied and added with the result being another natural number. However, if we consider \( 1 - 2 \) the result is \( -1 \) which is not a natural number. We will say that a set of numbers is closed under a given operation if combining two elements under the operation does not take us out of the given set of numbers. We can therefore say that \( \mathbb{N} \) is closed under addition and multiplication but not addition and division.

**Integers:** This is the set \( \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \) containing natural numbers, zero and negative whole numbers. \( \mathbb{Z} \) is closed under addition, subtraction and multiplication but not division.
Rational Numbers: The set of rational numbers denoted by \( \mathbb{Q} \) contains all numbers that can be written as a quotient of two integers. We can write \( \mathbb{Q} = \{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \} \). It is worth noting that the set of integers is a subset of the rationals. In our definition we have said that an element belonging to \( \mathbb{Q} \) can be written as a quotient of two integers and as such \( \pi \) is not a rational number. It is not correct to say that \( \pi \) can be written as \( \frac{22}{7} \) because the value of \( \pi \) is not equal to \( \frac{22}{7} \), however we are allowed to take \( \frac{22}{7} \) as an approximation to \( \pi \). The set \( \mathbb{Q} \) is closed under all the four basic operations of addition, subtraction, multiplication and division.

Irrational Numbers: These are numbers that cannot be written as a fraction of two integers. We may denote irrational numbers as \( \mathbb{I} \). They are nonrepeating, nonterminating decimals. \( \pi \), \( e \) and \( \sqrt{2} \) are irrational numbers. \( e \) has an approximate value of 2.718.

Real Numbers: This set is denoted by \( \mathbb{R} \) and is the union of the rational and irrational numbers. That is, \( \mathbb{R} = \mathbb{Q} \cup \mathbb{I} \). Observe that \( \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \). The set of real numbers is usually pictured as the set of all points on a line, as shown overleaf. The number 0 corresponds to a middle point, called the origin. A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin. Each point to the left of the origin corresponds to a negative real number, which is found by computing its distance from the origin and putting a minus sign in front of the resulting number. The set of real numbers can be divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0. Note that 0 is neither positive nor negative.

Figure 1.2

The Real Number Line

1.3.2 Basic arithmetic operations.

Basic arithmetic operations are:
(a) Addition (+)
(b) Subtraction (-),
(c) Division (÷),
(d) Multiplication (x)

1.3.3 BODMAS

An acronym derived from “Brackets Of Division, Multiplication, Addition and Subtraction”. In mathematics BODMAS is an order of operations or a rule used to unambiguously clarify which procedures should be performed first in a given mathematical expression.

Following the letters, the rule is that any items in Brackets should be combined first, division comes next then multiplication. Addition and subtraction should be performed last.
Example 3:

1) \[ 3 - 5 \times 4 \div 8 + 2 = 3 - 5 \times 0.5 + 2 = 3 - 2.5 + 2 = 2.5 \]

2) \[ (3 - 5) \times 4 \div 8 + 2 = -2 \times 4 \div 8 + 2 = -2 \times 0.5 + 2 = 1 \]

In 1 above, brackets may be used to avoid confusion, thus the expression may also be interpreted as:

\[ 3 - (5 \times 4 \div 8) + 2 = 3 - 2.5 + 2 = 2.5 \]

CHAPTER SUMMARY

In this chapter we have looked at the following:

- Mathematical notation.
- Mathematical symbols
- BODMAS

END OF CHAPTER EXERCISES

1. State the meanings of the following symbols: !, \sum, \leq, \pm.
2. Evaluate
   a) \[ -5 \times 4 \div 2 + 8 \]
   b) \[ -(5 \times 4) \div 2 + 8 \]
3. Evaluate
   a) \[ 7 \times 4 + \frac{7-1}{14} \]
   b) \[ 7 \times \left(4 + \frac{7-1}{14}\right) \]
CHAPTER 2  FRACTIONS, DECIMALS AND PERCENTAGES

LEARNING OBJECTIVES

At the end of this chapter the student should be able to.

i. Define a fraction and decimal
ii. Classify fractions
iii. Perform arithmetic operations on fractions and decimals
iv. Use them in a practical situation

2.0 INTRODUCTION

The term “fraction” refers to the concept of “part of a whole”. In practice measurements do not only concern whole entities. They may involve parts or “fractions” of the entity. A decimal is a fraction expressed as a part of a ten or power of ten. Thus fractions or decimals are an important aspect of measurement.

2.1 GENERAL EXPRESSION OF A FRACTION

A fraction is a part of an entity and generally a fraction is expressed as \( \frac{x}{y} \) where:

- \( x \) is referred to as the numerator
- \( y \) is referred to as the denominator. \( y \) cannot be zero.

Example 1

The following are some fractions:

\( \frac{1}{2}, \frac{3}{4}, \frac{7}{6}, \frac{1}{2} \)

2.2 TYPES OF FRACTIONS

It is possible to distinguish between 3 types of fractions. These are:

2.2.1 Proper fractions

For these fractions \( x < y \) where \( x \) is the numerator and \( y \) denominator.

Example 2

\( \frac{1}{2}, \frac{17}{19}, \frac{-3}{4}, \frac{7}{10} \), are all proper fractions.

2.2.2 Improper fractions

Generally they take the form \( \frac{x}{y} \) with \( x > y \).

Example 3

\( \frac{7}{2}, \frac{6}{4}, \frac{10}{7} \) are improper fractions

2.2.3 Mixed Numbers

This is where there is a whole number and a fraction.
Example 4 \[ 3 \frac{1}{2} \text{ and } 2 \frac{1}{3} \text{ are mixed numbers} \]

### 2.3 OPERATIONS INVOLVING FRACTIONS

Arithmetic operations on fractions include addition, subtraction, multiplication and addition.

**Example 5** Basic arithmetic operations:

a) **Addition:**
\[
\frac{1}{2} + \frac{3}{4} = \frac{2 + 3}{4} = \frac{5}{4} = 1 \frac{1}{4}
\]

b) **Subtraction:**
\[
\frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}
\]

c) **Multiplication:**
\[
\frac{3}{4} \times \frac{1}{2} = \frac{3 \times 1}{4 \times 2} = \frac{3}{8}
\]

d) **Division:**
\[
\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{3 \times 2}{4 \times 1} = \frac{6}{4} = \frac{3}{2} = 1 \frac{1}{2}
\]

### 2.4 DECIMALS

**2.4.1 Definition**

A decimal is a fraction whose denominator is ten or a power of ten. The following examples show simple decimals:

**Example 6** Write down any 4 simple decimals

Solution 0.5, 0.75, 0.003, 0.8

**2.4.2 Representation of Decimals**

A decimal (the fraction part) is preceded by a dot “.”.

From Example 6 we have 0.5, 0.75, etc.

It is possible to have a larger number which has a whole number and a fraction part like the case of a mixed number. The whole number will be shown followed by a dot and then the fraction part.

For example, 24.75. (This is equivalent to 24\% in terms of mixed numbers in normal fractions.

**2.4.3 Converting fractions to decimal**

A normal fraction or a mixed number can be converted to a fraction by dividing the numerator by the denominator. If the number to be converted is a mixed number, it must be turned into an improper fraction before conversion.

**Example 7** Convert a) \( \frac{1}{2} \) b) \( \frac{5}{8} \) c) \( 3 \frac{1}{4} \) d) \( 3 \frac{1}{7} \)
Solution

a) \[ \frac{1}{2} = 1 \div 2 = 0.5 \]

b) \[ \frac{5}{8} = 5 \div 8 = 0.625 \]

c) \[ 3 \frac{1}{4} = \frac{13}{4} = 13 \div 4 = 3.25 \]

d) \[ 3 \frac{1}{7} = \frac{22}{7} = 22 \div 7 = 3.143 \]

Note that when the decimal part goes on and on as in part d) above, rounding up to a required number of decimal places is used. See section 2.4.5 below.

2.4.4 Arithmetic operations on decimals
Like the case of fractions there are four basic arithmetic operations as shown below. While the calculations can be done using a calculator we also encourage you to know how to handle these calculations by hand.

Example 8:

a) Addition: \[ 2.492 + 34.124 \]

\[
\begin{array}{c}
2.492 \\
34.124 \\
\hline
36.616 \\
\end{array}
\]

c) Multiplication: \[ 5.2 \times 2.31 \]

\[
\begin{array}{c}
\phantom{0} 52 \\
\times \phantom{0} 231 \\
\hline
11550 \\
\end{array}
\]

Final number of decimals places = total decimal places in multiplicand and places in multiplier

d) Division: \[ 24.66 \div 1.2 \]

\[
\begin{array}{c}
\phantom{0} 24.66 \\
\div \phantom{0} 1.2 \\
\hline
22.55 \\
\end{array}
\]

2.4.5 Rounding

Often times figures may have too many digits when a few (shorter string of numbers) are enough to give a good picture of the value measured. Rounding is that technique used to reduce the number of digits used to a desired few.

Common forms of rounding are:

(a) to the nearest whole number
(b) to a number of decimal places
(c) to a number of significant figures

Rules for Rounding

(a) If the number you are rounding is followed by 5, 6, 7, 8, or 9, round the number up.
(b) If the number you are rounding is followed by 0, 1, 2, 3, or 4, round the number down.

Significant Figures
The number of significant digits in an answer to a calculation will depend on the number of significant digits in the given data. Non-zero digits are always significant. Thus, 26 has two significant digits, and 27.9 has three significant digits.

With zeroes, the situation is more complicated:
- a. Zeroes placed before other digits are not significant; 0.078 has two significant digits.
- b. Zeroes placed between other digits are always significant; 609 has three significant digits.
- c. Zeroes placed after other digits but behind a decimal point are significant; 8.10 has three significant digits.
- d. Zeroes at the end of a number are significant only if they are behind a decimal point as in (c). Otherwise, it is impossible to tell if they are significant. For example, in the number 8200, it is not clear if the zeroes are significant or not. The number of significant digits in 8200 is at least two, but could be three or four.

We are now ready to look at how one can round a number to a given number of significant figures. Generally, rounding a number to \(n\) significant figures works the same way as rounding. If the first non-significant figure is a 5, 6, 7, 8, or 9 round up the last significant figure else round down.

**Example 9**

(a) Round the following to the nearest whole number:
- i) 47.25 = 47
- ii) 47.52 = 48
- iii) 100.789 = 101
- iv) 79.49 = 79
- v) 1999.51 = 2000

(b) Round the following to the number of decimal places indicated:
- i) 47.257 to 2 decimal places = 47.26
- ii) 47.529 to 2 decimal places = 47.53
- iii) 100.98 to 1 decimal place = 101.0
- iv) 1999.59 to 1 decimal place = 1999.6

(c) Rounding to a number of significant figures
- i) 47.257 to 3 significant figures = 47.3
- ii) 47.529 to 2 significant figures = 48
- iii) 100.998 to 1 significant figure = 100
- iv) 1999.56 to 3 significant figures = 2000

### 2.5 PERCENTAGES

A percentage is a way of expressing a number as a fraction of 100. The word per cent means per hundred. The denominator “100” is not shown but it is denoted by the percent sign “%”
Example 10

a) The fraction \(\frac{1}{2}\) can be expressed as \(\frac{50}{100}\) which is 50%

b) \(\frac{4}{10}\) can be expressed as \(\frac{40}{100}\) which is 40%

2.5.1 Converting Fractions and Decimals to percentages

Converting Fractions to percentage

The process is simply to multiply the fraction with 100 and reduce to lowest terms.

Example 11  Convert the following fractions to percentages

i) \(\frac{3}{4}\) ii) \(\frac{35}{80}\) iii) \(\frac{30}{25}\)

Solution

i) \(\frac{3}{4} = \frac{3}{4} \times 100 = 75\%\)

ii) \(\frac{35}{80} = \frac{35}{80} \times 100 = 43.75\%\)

iii) \(\frac{30}{25} = \frac{30}{25} \times 100 = 120\%\)

Converting decimals to Percentages

Just like in the case of fractions, to convert a decimal to a percentage the decimal is multiplied by 100.

Example 12

Convert the following to percentages

i) 0.34 ii) 0.78 iii) 0.8

Solution

i) 0.34 to percentage: 0.34 \(\times 100 = 34\%\)

ii) 0.78 to percentage: 0.78 \(\times 100 = 78\%\)

iii) 0.8 to percentage: 0.8 \(\times 100 = 80\%\)
2.5.3 Arithmetic operations on percentages

Addition and subtraction

As far as simple arithmetic operations are concerned percentages can be added or subtracted in the same way as integers or normal decimals are added or subtracted.

Example 13

i) \[ 30\% + 45\% = 75\% \]

ii) \[ 57.3\% + 25.2\% = 82.5\% \]

iii) \[ 85\% - 22.4\% = 62.6\% \]

Multiplication and division

Percentages can be multiplied or divided just like fractions and decimals can be multiplied or divided. However rules governing the multiplication and division of fractions and decimals apply.

Example 14

i) \[ 20\% \times 50\% \text{ (which is really } 20\% \text{ of } 50\% \text{) } = 0.2 \times 0.5 = 0.1 = 10\% \]

ii) \[ 20\% \div 50\% = 0.2 \div 0.5 = 0.4 = 40\% \]

\[
\text{OR}
\]

\[ 20\% = \frac{20}{100} \text{ and } 50\% = \frac{50}{100} \]

\[ \text{Therefore } \frac{20\%}{50\%} = \frac{\frac{20}{100}}{\frac{50}{100}} = \frac{20}{50} \times \frac{100}{100} = \frac{2}{5} = 0.4 = 40\% \]

CHAPTER SUMMARY

In this chapter we have looked at the following:

- The difference between proper and improper fractions.
- Addition, subtraction, multiplication and division of fractions.
- Addition, subtraction, multiplication and division of decimals.
- Rounding off a number to a given number of decimal places.
- Rounding off a number to a given number of significant figures.
- Converting fractions and decimals to percentages.
- Arithmetic operations on percentages.
END OF CHAPTER EXERCISES

1. Arrange the following in ascending order: \( \frac{2}{7}, \frac{3}{8}, \frac{4}{10} \) and \( \frac{5}{12} \)

2. Evaluate: \( \frac{1}{2} + \frac{1}{3} + \frac{2}{5} \)

3. Find a) \( \frac{2}{5} - \frac{4}{7} \) b) \( \frac{2}{5} \div \frac{4}{7} \)

4. What is 20% of 20½

5. Simplify
   a) \( \frac{2}{3 + \frac{4}{2 + \frac{2}{3}}} \)
   b) \( \frac{\frac{17}{3} + (\frac{2}{4} + \frac{3}{4})}{\frac{3}{14}} \)

6. A man uses 1/8 of his salary after tax on house rent and a 3/5 on transport, food and other household items. He then reserves 5/8 of the remainder on leisure and incidentals while saving the rest. If he saves K37,135 every month.
   a) How much is his Salary after tax?
   b) How much is the rent per month?
   c) What is the percentage of the salary after tax spent on leisure and incidentals?

7. If the inlet pipe for a tank is opened, the tank will be filled in 6 hours. If the outlet pipe for a tank is opened when the tank is full, the tank will be emptied in 8 hours. How long will it take to fill the whole tank if both the inlet and outlet pipes are open?
CHAPTER 3 RATIOS AND PROPORTIONS

LEARNING OBJECTIVES

By the end of this chapter the student should be able to:

i) define a ratio and a proportion
ii) to formulate and apply ratios and proportions
iii) understand the concept of proportional parts and solve associated problems

3.0 INTRODUCTION

Both ratios and proportions refer to a comparison between quantities. Ratios and proportions are used in everyday life because in most dealings one has to make comparisons between entities.

3.1 RATIOS

3.1.1 Definition

A ratio is a comparison of two quantities. The items that they relate to may or may not be related.

Example 1 A mathematics class has 12 female and 15 male students. What is the ratio of female to male students.

Solution The ratio of female to male students is 12 to 15 written as 12:15.

3.1.2 Expression of ratios

As indicated above a ratio is expressed as two numbers separated by colon.

Example 2 There are 10 goats and 25 cattle in a feeding pen. Express the number of goats and cattle as a ratio.

Solution The ratio in this case of goats to cattle is 10:25

Note that a ratio can be expressed as a fraction.

Example 3 Express the ratios in examples 1 and 2 above as fractions

Solution The ratio in example 1 is 12:15  This can be written as 12/15

The ratio in example 2 is 10:25  This is 10/25

Further it is common to express ratios to their lowest terms

Example 4 Express the ratio 12:15 in its lowest terms

Solution Divide both by 3-(the highest common factor).
The result is 12:15 = 4:5

Example 5  The ratio \( \frac{10}{15} = \frac{2}{3} \)

Example 6  A concentrated chemical used to treat termites on a building site has to be diluted as follows: 1½ litres of the chemical to 30 litres of water. Express the relative quantities of the chemical and water as a ratio.

Solution  The ratio is 1½ to 30 = 3/2 to 30
= 6 : 60
= 1:10 to the lowest terms

3.1.3 Application examples

Example 7  A Toyota Corolla goes 15KM on one litre of petrol. How much petrol will it need to get to Kasungu from Lilongwe, Kasungu being 120KM away.

Solution  The ratio of distance to fuel here is 15:1

Let the fuel required be \( x \)

Therefore 15 : 1 must equal 120 : \( x \)

15 has increase by a factor 120/15

1 must increase by the same factor: Fuel required is \( \frac{120}{15} \times 1 = 8 \) lt

Example 8  Two people share proceeds of a business venture in the ratio 2:3 If the proceeds came to MK2,350,000 how much does each get?

Solution  Let A and B represent the persons sharing the proceeds

Total parts from the ratio: 2 + 3 = 5

A therefore will receive \( \frac{2}{5} \times 2,350,000 = \) MK94,000

B will receive \( \frac{3}{5} \times 2,350,000 = \) MK1,410,000

Example 9  KAWIYA Tea Estate produces a high quality tea branded Kawiya by blending three types of tea coded A, B, and C in the ratio 1½ : 5 : 1. Originally Type A tea costs MK1,600 type B costs MK800 and type C costs MK1,700 per Kg to produce. Kawiya Tea Estate packs Kawiya tea in packets of 825g each. Blending and packing costs are 40 per Kg.
Required:
Determine the production cost for a packet of Kawiya tea.

Solution
Ratio of the teas in the blend A:B:C = 1½ : 5 : 1
= 3 : 10 : 2
Total parts = 3 + 10 + 2 = 15

Amount of tea A in the packet = $825 \times \frac{3}{15} = 165g$

Tea B: $825 \times \frac{10}{15} = 550g$

Tea C: $825 \times \frac{2}{15} = 110g$

Contribution in costs:
A: $165 \times K1.60 = 264$
B: $550 \times K0.80 = 440$
C: $110 \times K1.70 = 187$

Total cost of the teas in the packet = $891$
Blending and packaging = $40$

Total production cost per packet = MK931

Example 10
Two brothers, Mayeso and Khoza earn K80,000 and K150,000 respectively as salaries per month. The brothers come from a saving culture and Mayeso saves K25,000 while Khoza saves K30,000 per month. Who between the two brothers has a greater savings to salary ratio?

Solution
Let “e” and “s” represent earnings and savings respectively

For Mayeso:
$s : e = \frac{25000}{80000} = \frac{5}{16}$

For Khoza:
$s : e = \frac{30000}{150000} = \frac{1}{5}$

Expressing the ratios to a common denominator:

$\frac{5}{16} = \frac{25}{80}$ and $\frac{1}{5} = \frac{16}{80}$

Example 11
Mayeso has a greater savings to salary ratio. (note that percentages could have been used)
The rent for the house in which Mr. Chikumbu lives has been increased in the ratio 4:3. If the old rent was K60,000 per month. Find the new rent.

Solution
Let $X = \text{new rent}$.

$4 : 3 = X : 50,000$ or $\frac{4}{3} = \frac{X}{60000}$

$X = 60,000 \times \frac{4}{3}$

$= K80,000$
Example 12  Akwawo, Bvuto and Chatha form a business. They agree to share profits in the ratio 4:3:1 respectively and at the end of the year they make total sales of K457,820. If the total cost was K250,000 how much does each receive?

Solution  
Profit = Revenue - Cost  
\[ 457,820 - 250,000 = 207,820 \]

Ratio for sharing the profit: 4:3:1  
Total parts = 4+3+1 = 8

Akwawo gets  
\[ \frac{4}{8} \times 207,820 = K103,910.00 \]

Bvuto gets  
\[ \frac{3}{8} \times 207,820 = K77,932.50 \]

Chatha gets  
\[ \frac{1}{8} \times 207,820 = K25,977.50 \]

3.2 PROPORTIONS

3.2.1 Definition
A proportion is a relationship between 2 parts of an entity. Emphasis here is on the fact that the parts would be related. For example, a car which is supposed to travel a distance of 400 km only does 300 km. The part it travelled can be expressed as a proportion of the entire distance.

Proportions are based on the concept of ratios (direct proportionality):

Example 13  The Business Mathematics class at your college has 20 female students and 30 male students. Find

a) The ratio of female to male students.
b) The proportion of female students in the class.

Solution  
a) Female : Male  
\[ 20 : 30 = 2 : 3 \]

b) Female students/total class  
\[ \frac{20}{20 + 30} = \frac{20}{50} = \frac{2}{5} \text{ or } 40\% \]

Example 14  Chikhosi has just started work and just clocked 12 days to the end of the first month. As per regulation his pay is to be prorated on days. His monthly salary is MK96,000 and assuming a 30-day month, calculate his salary for the first month.

Solution  
Proportion of days on the job: 12/30

Pay for the month  
\[ 96,000 \times \frac{12}{30} = 3200 \times 12 = K38,400 \]

Example 15  James can travel a distance of 10 km in 2 hours. How long would he take to cover 16 km.
Solution \[
\frac{16}{10} \times 2 = 3\text{hrs 12min} \quad (1 \text{hr} = 60 \text{min} \text{ so } .2 \text{ hrs} = 12\text{min}).
\]

### 3.2.2 Proportional Parts

The concept of proportional parts is used to link two or more relationships (ratios) in order to relate the elements in the different sets of ratios.

**Example 16**

Given \(a:b = 3:2\) and \(b:c = 3:5\) Find \(a:b:c\)

**Solution:**

Step 1 Determine the common element. This is \(b\)

Step 2 Make the common element have the same value in each ratio. As a guide use the lowest common multiple to calculate this value.

In this case we have \(2 \times 3 = 6\)

The ratios can be: \(a:b = ? : 6\) and \(b:c : 6: ?\)

Step 3 In each ratio the other elements are multiplied or divided by the factor by which the common element \(b\) has been changed.

In \(a:b\), \(b\) has been multiplied by 3 therefore element \(a\) is multiplied by 3

\[3 \times 3 = 9. \quad \text{Thus } a:b = 9:6\]

For \(b:c\), \(b\) was multiplied by 2 to get to 6, therefore \(c\) should be multiplied by 2.

\[5 \times 2 = 10 \quad \text{Thus } b:c = 6:10\]

\[a:b = 9:6 \quad b:c = 6:10 \quad a:b:c = 9:6:10\]

**Example 17**

The ages of George and his sister Helen are in the ratio 3:2. The ratio of Georges age to that of his friend Peter is 7:8. If George’s sister is 21 years old, how old is Peter?

**Solution:**

The ratios are \(G:H = 3:2, \ G:P = 7:9\)

We need: \(G:H:P\)

Common element = \(G\)

The lowest common multiple is \(3 \times 7 = 21\)

Therefore \(G:H = 3:2 = 21:14, \ (3:2 \text{ multiplied by } 7)\)

\(G:P = 7:9 = 21:27 \ (7:9 \text{ multiplied by } 3)\)

\[G:H:P = 21:14:27 \quad \text{Helen is 28years old}\]


\[21/14 = 1.5 \quad \text{All other elements should be multiplied by 1.5}\]

\[\text{Peter’s age} = 27 \times 1.5 = 40.5 \text{ year}\]
CHAPTER SUMMARY

In this chapter you have learned the following:

i. Definition of ratio and proportion.

ii. Formulated problems involving proportions

iii. Worked out practical examples on ratios and proportions.

iv. Applied the concept of proportional parts and solved associated problems.

END OF CHAPTER EXERCISES

1. Two types of tea costing K650 per kilogram and K720 per kilogram respectively are blended in the ratio 7:3. Calculate the cost of 1 kilogram of the blended tea.

2. A prudent farmer mixes fertiliser 25Kg of fertiliser A and 40Kg of fertiliser B per half acre of a particular crop. Find the ratio of the two fertilisers in the mixture.

3. Three club employees agree to share their Christmas bonus in the ratios of their ages which are 45 years, 48 years and 51 years. The total bonus is K67,200. How much does each employee receive?

4. Three persons contribute 200000, 300000 and 150000 to start a business. At the end of the year their sales amount to K2,450,000 with costs of K505,000. They discover they do not qualify for corporation tax but under the regime they are supposed to pay a tax calculated as 10% of revenue. If they are to share the net profit according to their contributions, how much does each get?

5. If $a:b = 3:2$ and $b:c = 3:4$ Find $a:b:c$;

6. Consider the example of Akwawo, Bvuto and Chatha, if Akwawo is to receive $1\frac{1}{2}$ as much as Bvuto gets and Chatha gets $\frac{1}{2}$ as much as Bvuto calculate what each receives if no taxes are parables.

7. Titha Construction company is pulling down a building to pave way for a new one and it has two teams that do the job. Team A can pull the building down and clear the site in 3 weeks. Team X can do the job in 2 days. How long will it take them if they work together?
CHAPTER 4 FUNCTIONS

OBJECTIVES

By the end of this chapter students should be able to:

i) Define a function.
ii) Distinguish the various types of functions.
iii) Formulate equations.
iv) Solve linear equations.
v) Solve quadratic equations by graph, factorisation and formula.
vi) Solve exponential and logarithmic equations.
vii) Solve compound interest problems using logarithms.

4.0 INTRODUCTION

This chapter seeks to introduce you to the very important concept of a function. The notion of a function is important in that it gives a formula which relates seemingly (in some cases) unrelated objects. With a formula in hand, one may be able to determine how a change in one variable affects the other variable. The chapter will also introduce you to some business applications of functions.

4.1 FUNCTIONS IN GENERAL

4.1.1 Definition of a function.

In a mathematical context, a function is a relationship between variables. It relates an input or a set of input to a unique output. For example, if we take a function which takes an input and squares it to give the output, then 4 is the output if we input 2. We can think of a function as a machine which takes the input, processes it and gives an output. In an algebraic function, one variable depends on or is influenced by other variables.

4.1.2 Notation of a function

If \( f(x) \) is input into a function then it's output will be denoted \( f(x) \). \( f(x) \) is read as ‘ef of x’. We will also write \( y = f(x) \) if \( y \) depends on \( x \) or \( y \) is a function of \( x \). We call \( f(x) \) the image of \( x \) under \( f \). Note that we will also write a function \( f(x) \) as \( f \).

Example 1  \( f(x) = 3x + 10 \) is a function. It takes the input \( x \) multiplies it by 3 and then adds 10.

Example 2  \( G = 3k + 4 \), is a function which can be written as \( G = f(k) \).

Example 3  Given \( K = x^2 + 8y \), \( K \) is a function of \( x \) and \( y \).

Example 4  \( M = x^{1/2} \) is not a function since if \( x = 4 \) then \( M = \pm 2 \) and we know that each input has to correspond to a unique output.

4.1.3 Domain and Range

A function's input is called the domain, and its output is called the range.
Example 5  Consider the diagram below:

![Figure 4.1 domain of a function](image)

It is clear from the diagram that the domain of the function depicted is \([-1, 0, 2, 4]\) and the range is the set \([1, 4, 9]\).

The domain of a function can be any set of numbers which have images. This could be a few numbers or the whole set of real numbers. In cases where some elements have no images, such elements must be excluded from the domain.

Example 6  State the values of \(x\) that must be excluded from the domain of the following functions

a. \(f(x) = \frac{1}{x}\)

b. \(g(x) = 3 - \frac{1}{x+3}\)

Solution:

a. Every real number will have an image under this function except \(x = 0\) because the function is undefined when \(x = 0\). So the domain of this function is the set \([all \ real \ numbers \ x : x \neq 0]\).

b. We begin by writing the expression with a common denominator.

Doing this gives \(g(x) = \frac{3(x+3)-1}{x+3} = \frac{3x+8}{x+3}\). It is not difficult to see that the function is undefined if we input \(x = -3\) (look at the denominator!).

As such the domain of this function is the set of all real numbers except \(x = -3\).

Example 7  State the range of \(f(x) = x^2\).

Solution:  Here the domain is the set of all real numbers. Now, if we input any real number whether negative or positive we see that the output will be positive as such the range is the set of all positive real numbers.
4.2 EVALUATING FUNCTIONS

In this section we will look at how to evaluate functions. Let us do some examples:

Example 8 If \( f(x) = x^2 + 2x - 1 \), find \( f(1) \).

Solution: All we need to do here is put 1 where ever there is \( x \). Doing that gives \( f(1) = 1^2 + 2(1) - 1 = 2 \)

Note: To evaluate \( y = f(x) \), everywhere we see an \( x \) on the right side we will substitute whatever is in the parenthesis on the left side.

Example 9 Given \( f(x) = -x^2 + 6x - 11 \), find each of the following.

a) \( f(2) \)

b) \( f(t) \)

c) \( f(x - 3) \)

Solution

a) \( f(2) = -2^2 + 6(2) - 11 = -4 + 12 - 11 = -3 \).

b) Remember that we substitute for the \( x \)’s WHATEVER is in the parenthesis on the left. Often this will be something other than a number. So, in this case we put \( t \)’s in for all the \( x \)’s on the left.

This gives \( f(t) = -t^2 + 6t - 11 \).

c) To find \( f(x - 3) \) we evaluate the given function at \( x - 3 \), that is we take the input as \( x - 3 \).

\[
f(x - 3) = -(x - 3)^2 + 6(x - 3) - 11 = -(x^2 - 6x + 9) + 6x - 18 - 11 = -x^2 + 12x - 38.
\]

4.3 COMBINING FUNCTIONS

Functions may be added, multiplied, subtracted and divided to form new functions. In general we will use the following:

Let \( f(x) \) and \( g(x) \) be functions then;

1. \( (f \pm g)(x) = f(x) \pm g(x) \),
2. \( (fg)(x) = f(x)g(x) \) is the product between \( f(x) \) and \( g(x) \) and
3. \( \frac{f}{g}(x) = \frac{f(x)}{g(x)} \), provided \( g(x) \neq 0 \).

Example 10 Given \( f(x) = 2 + 3x - x^2 \) and \( g(x) = 2x - 1 \), evaluate each of the following

a) \( (f + g)(4) \)

b) \( (g - f)(x) \)

c) \( (fg)(x) \)

d) \( \frac{f}{g}(0) \).
Solution: We will do a) and d) and leave b) and c) to the reader.

\[(f + g)(4) = f(4) + g(4) = (2 + 3(4) - 4^2) + (2(4) - 1) = 5.\]
\[\frac{f}{g}(0) = \frac{f(0)}{g(0)} = \frac{2}{-1} = -2.\]

4.4 COMPOSITE FUNCTIONS
Consider the function \(f(x) = 2x + 4\). This function maps 4 onto 12, that is, \(f(4) = 12\).
Let \(g(x)\) be another function such that \(g(x) = x + 1\). We now apply \(g\) to \(f(4)\) to obtain \(g(f(4)) = g(12) = 13\).
So 4 has been mapped onto 13 by \(f\) followed by \(g\). The question is, can we find a single function \(h\) which combines \(f(x)\) and \(g(x)\)?

In the example above, \(x\) is mapped onto \(2x + 4\) by \(f\) and this is what we plug into \(g\). So \(g\) maps \(2x + 4\) onto \((2x + 4) + 1 = 2x + 5\).

Hence \(h(x) = 2x + 5\). If \(x = 4\) the final result is \(h(x) = 2(4) + 5 = 13\).
\(h\) is called the composite function \(g(f(x))\) which we write as \(h = g(f(x)) = g \circ f(x)\).
Note that in \(g \circ f(x)\) we apply \(f(x)\) first and \(g\) second.

It is up to reader to confirm that generally
\(g \circ f(x) \neq f \circ g(x)\) using \(f(x) = 2x + 4\) and \(g(x) = x + 1\).
Hence as far as the composition of functions is concerned order is very important.

Example 11 If \(f(x) = x^2\) and \(g(x) = x - 1\), find \(f \circ g\) and \(g \circ f\).

Solution: \(f \circ g = f(g(x)) = f(x - 1) = (x - 1)^2\) and \(g \circ f = g(f(x)) = g(x^2) = x^2 - 1\).

4.5 TYPES OF FUNCTIONS
There are many types of functions but of interest to us are linear, quadratic, logarithmic and exponential functions.

4.5.1 LINEAR FUNCTIONS

a) General form of Linear Functions

In general linear functions are functions which have variables with highest power equal to one. A general linear function takes the form \(f(x) = mx + c\) where \(m\) and \(c\) are constants and \(m\) is nonzero.

Example 12 Which of the following are linear functions?

a. \(y = 15x + 10\)
b. \(y = x\)
c. \(k = 2r + 3\)
d. \(y = \frac{1}{x}\)
e. \(2x = 3 - 5y\)
f. \(y = x^2 + 3x - 4\)
Solution: a), b), c), and e) are linear equations. d) and f) are not linear equations.

b) Linear Equations

A linear equation is derived from a linear function. The general format for a linear equation is therefore \( y = mx + c \) where \( m \neq 0 \) and \( c \) are constants.

c) Equations from some scenario

Sometimes equations can be presented in a form ready to be solved. At other times one needs to formulate the equation from a context given.

Example 14

A manufacturer pays K25,000, on material and other variable inputs to produce one unit of a product DS20. If rent and other fixed cost amount to K120,000 per month. Find an expression of the total cost of making any quantity of the product.

Solution.

Step 1 Determine items that will vary (the variables). These are quantity and costs and let these be \( x \) and \( y \)

Where \( x = \) quantity produced
\( y = \) Total cost

Step 2 Note and write out the way in which variables change

From the text, variable cost and quantity produced are linearly related.
For every unit of \( x \) produced the variables cost is 25,000.
Therefore if \( x \) are produced the total variable cost is 25000x

Fixed costs do not change.

Step 3 Write out the expression.

Total cost is an addition of the variable and fixed elements.

Therefore total cost = Fixed cost + variable cost
= 120000 + 2500x

This is a linear equation of the form \( y = mx + c \).

d) The Graph of a linear function

The graph of a linear function \( y = f(x) = mx + c \) is straight line. There are several ways of drawing the graph of a linear function:
i. **Using a table of values:** In this method we create a table to hold $x$ and $y$ values (coordinates) from the equation. We choose $x$ values according to an appropriate scale, plug the $x$-values into the given function to obtain corresponding values of $y$. The $x$ and $y$ values are plotted on a graph and then a line is drawn to pass through the plotted points.

ii. **Plotting the $x$ – and $y$ – intercepts:** Here we focus on points where we expect the line to cut the $x$ and $y$ axes called the $x$ – and $y$ – intercepts respectively. The $x$ – intercept is found by setting $y = 0$ and solving the equation for $x$. To find the $y$ – intercept we set $x = 0$ and solve the equation for $y$. The graph is found by drawing a line through the $x$ and $y$ intercepts.

**Example 15:** Draw the graph of $y = 2x + 3$.

**Solution**

First determine the range of $x$ values according to need. For this example $x$ values from -4 to +6 are adequate. For simplicity the $x$ values should be integers. Substitute the values into the equation to come up with the table as shown below. This technique is also useful when graphing non-linear equations.

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**Figure 4.2 Straight line graph**

<table>
<thead>
<tr>
<th>Coordinates:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
</tr>
<tr>
<td>$Y$</td>
</tr>
</tbody>
</table>

Note that the graph is a straight line (all points are on the line).
Example 16: Plot the graph of $6x + y = 12$.

Solution: We will sketch the graph using the $x -$ and $y -$ intercepts.

Figure 4.3 Graph from two coordinates

Note the direction of the line. It is going downwards from left to right. This is a negative slope. The first was a positive gradient.

**e) The slope of a straight line graph**

The slope of a line, also called the gradient, is a measure of the steepness of the line. Generally, if the slope is positive the line moves upward as it moves to the right and if the slope is negative the line moves downward as it moves to the right. This gives us a way of knowing whether the slope of a line is positive or negative if we are given its graph. The slope of a line can be deduced from the general form of the given equation. If we write a linear function in the form $y = mx + c$, then $m$ is the slope of the line and $c$ is the $y -$ intercept.

Alternatively, the slope can be calculated from any two coordinates on a graph. The slope is the ratio of a change in the $y$ coordinate to the corresponding change in the $x$ coordinate. This can be written as

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$

or, equivalently, as $\frac{\Delta y}{\Delta x}$.

Given the coordinates as $(x_1, y_1)$ and $(x_2, y_2)$ then the slope is given by $\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$.

Example 17: Given the following linear graph, calculate the slope.
Figure 4.4  Slope of a graph

We have the two points (2,6) and (6,10), therefore \((x_1, y_1) = (2,6)\) which means \(x_1 = 2\) and \(y_1 = 2\).
Also, \((x_2, y_2) = (6,10)\) which means \(x_2 = 6\) and \(y_2 = 10\). So we have;

\[
\text{Slope} = \frac{y_2-y_1}{x_2-x_1} = \frac{10-6}{6-2} = \frac{4}{4} = 1. 
\]

f)  **Finding the equation of a line**

To find the equation of a line we need to know either a point through which the line passes and its gradient or two points on it.

g)  **Equation of a line given a point and gradient**

The equation of a line through \((x_1, y_1)\) and having gradient \(m\) is given by \( \frac{y-y_1}{x-x_1} = m \). This is equivalent to \(y - y_1 = m(x - x_1)\) called the point-slope equation of the line.
h) Equation of a line through two given points

The equation of a line through \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[
\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}
\]

You will notice that in this equation \(\frac{y_2 - y_1}{x_2 - x_1}\) is an expression for the gradient of the line. This tells us that even if we are given two points through which a line passes we can use the point-slope formula to find its gradient. We will demonstrate this in the example below.

Example 18 Given that a linear equation graph passes through the points \((-3,0)\) and \((2,7)\) find its equation.

Solution

Method 1 The equation takes the form \(y - y_1 = m(x - x_1)\) where slope \(= m\).

Let us calculate \(m, m = \frac{7-0}{2-(-3)} = \frac{7}{5}\). So the equation is \(y - 0 = \frac{7}{5}(x - (-3))\) which is \(5y = 7x + 21\).

Method 2 Remember that the equation of a line through \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[
\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}
\]

Here we have the two points \((-3,0)\) and \((2,7)\) so the equation is \(\frac{y - 0}{x - (-3)} = \frac{7-0}{2-(-3)} = \frac{7}{5}\).

That is, \(\frac{y}{x+3} = \frac{7}{5}\).

Simplifying this we get \(5y = 7x + 21\).

4.5. 2 LINEAR SIMULTANEOUS EQUATIONS

a) Definition

Simultaneous equations are equations which must be satisfied with one set of solutions. Such equations have two or more variables in the equations.

The following are examples of linear simultaneous equations (also referred to as systems of linear equations).

a. \(y = 4x + 2\) and \(y = 3 - 2x\)

b. \(2x + 3y = 11\) and \(x - y = -2\)

c. \(2x + 3y - z = 5, x - y + 2z = 5\) and \(3x + 2y + z = 10\)

b) Solving simultaneous equations

Methods of solving simultaneous equations

i. Graphical technique

ii. Elimination method

iii. Substitution method

c) The Graphical technique

For purposes of the level addressed by this text the graphical method is suitable for systems of equations with two variables and two equations. To determine the solution:
i. Draw the linear equations on the same axes
ii. Note the values of $x$ and $y$ where the graphs intersect. These values are the solution.

**Example 19**
Solve the following system of equations using the graphical technique: $y = 2x + 1$ and $y = 9 - 2x$.

**Solution:** We will use $x-$ and $y-$intercepts to plot the graphs.
For $y = 2x + 1$, when $x = 0$, $y = 1$ and when $y = 0$, $x = -1/2$. Therefore the line passes through the points $(0,1)$ and $(-1/2,0)$.

For $y = 9 - 2x$, when $x = 0$, $y = 9$ and when $y = 0$, $x = 9/2$. Therefore the line passes through the points $(0,9)$ and $(9/2,0)$.

Plotting the coordinates we have the following graph

![Graphical solution of simultaneous equations](image)

From the graph $x = 2$ and $y = 5$ (point of intersection)

**d) Using the elimination method**

In this method we continuously eliminate one variable by making suitable manipulations so that one equation in one unknown remains.

**Example 20** Given $y = 2x + 1$

$y = -2x + 9$

Solution
Number the equations as follows:

\[ y = 2x + 1 \quad \text{equation (1)} \]
\[ y = -2x + 9 \quad \text{equation (2)} \]

Note that the two equations have the coefficients on \( y \) and \( x \) similar. The variable \( x \) can be eliminated by adding (it has opposite signs).

Thus equation (1) + (2):

\[ 2y = 10 \quad \text{therefore} \quad y = 10/2 = 5 \]

using equation (1)

\[ 5 = 2x + 1 \]
\[ 2x = 4 \]
\[ x = 4/2 = 2 \]

\( x = 2 \) and \( y = 5 \)

**Example 21**  Find \( x \) and \( y \) using the elimination method given

\[ 2x + 3y = 11 \]
\[ x - y = -2 \]

Solution

\[ 2x + 3y = 11 \quad (1) \]
\[ x - y = -2 \quad (2) \]

Since the coefficients on \( x \) and \( y \) are not the same, make one of them the same:

Multiply equation (2) by 3 and add to eliminate \( y \):

\[ 2x + 3y = 11 \]
\[ 3x - 3y = -6 \]

The sum \( 5x + 0 = 5 \) therefore \( x = 5/5 = 1 \)

Substitute in (2) (any of the equations can do but this is simpler):

\[ 1 - y = -2 \quad \text{y = 3} \]

Therefore \( x = 1 \) and \( y = 3 \)

e) **Using the substitution method**

Here we solve one equation for one variable in terms of the other. We then substitute this expression into the other equations to determine the value of the first variable (if possible). Then substitute this value to determine the value of the other variable.

**Example 22**  Consider the same problem as above and solve for \( x \) and using the using substitution:

Solution

\[ 2x + 3y = 11 \quad (1) \]
\[ x - y = -2 \quad (2) \]
From equation (2) \[ x = y - 2 \quad \text{(3)} \]
x into equation (1): \[ 2(y - 2) + 3y = 11 \]
\[ 2y - 4 + 3y = 11 \]
\[ 5y = 15 \]
\[ y = 3 \]

Using (3) again \[ x = y - 2 \]
\[ = 3 - 2 \]
\[ x = 1 \]

**Example 23** The price of the particular product is set at 12 kwacha per item for ten (10) items and falls to K8.50 per item when 30 are ordered. Derive the Demand function. Assuming it is linear

**Solution:** Let \( p = \text{price} \) and \( q = \text{quantity} \)

As a linear function demand should be

\[ p = a + bq \]

The data provided is

<table>
<thead>
<tr>
<th>( q )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>K12</td>
</tr>
<tr>
<td>30</td>
<td>K8.50</td>
</tr>
</tbody>
</table>

These can be used to form two equations of the form

\[ 12 = a + 10b \quad \text{equation 1} \]
\[ 8.5 = a + 30b \quad \text{equation 2} \]

From equation 1 we have

\[ a = 12 - 10b \quad \text{equation 3} \]

Equation 3 into 2

\[ 8.5 = 12 - 10b + 30b \]
\[ -3.5 = 20b \]
\[ -3.5/20 = b \]
\[ -0.175 = b \]

Substitute \( b \) into equation 3

\[ a = 12 - 10 \times (-0.175) \]
\[ = 13.75 \]

The demand function: \[ p = 13.75 - 0.175q \]
f) **System of equations with 3 unknowns and 3 equations**

A system of more than two equations can be solved by using either the substitution method or the elimination method.

**Example 24**

Solve the following for x, y and z

\[
4x + y + 2z = 12 \\
3x - 2y + 2z = 5 \\
5x - y - z = 0
\]

**Solution:** Let the equations be

\[
4x + y + 2z = 12 \quad (i) \\
3x - 2y + 2z = 5 \quad (ii) \\
5x - y - z = 0 \quad (iii)
\]

Using equation (iii) make y the subject.

\[
-y = z - 5x \\
y = 5x - z \quad (iv)
\]

Substitute (iv) into (i)

\[
4x + (5x - z) + 2z = 12 \\
9x + z = 12 \quad (v)
\]

Substitute (iv) into (ii)

\[
3x - 2(5x - 2) + 2z = 5 \\
-7x + 4z = 5 \quad (vi)
\]

We can now solve (v) and (vi) for x and z having eliminated y

\[
9x + z = 12 \quad (v) \\
-7x + 4z = 5 \quad (vi)
\]

Making z the subject in (v):

\[
z = 12 - 9x \quad (v)
\]

Substitute into (vi)

\[
-7x + 4(12 - 9x) = 5 \\
-7x + 48 - 36x = 5 \\
-43x = 5 - 48 \\
-43x = -43 \\
x = 1
\]

Therefore, \( x = 1 \)
Using equation (iv) \[ y = 5(1) - 3 = 2 \]

Therefore \( x, y, z = 1, 2, 3 \)

**Example 25**  Kupanga ltd makes 3 types of shoes code named X241, Youthquake and Zachangu. Each of the products pass through three machines in their manufacture. The time in hours each unit takes in a particular machine is given below:

<table>
<thead>
<tr>
<th>Product</th>
<th>Time spent in Machine</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>X241</td>
<td>3</td>
</tr>
<tr>
<td>Youthquake</td>
<td>3</td>
</tr>
<tr>
<td>Zachangu</td>
<td>2</td>
</tr>
</tbody>
</table>

There 130 hours, 85 hours, and 60 hours of machines A, B, and C respectively, available in a week. If the aim is to use up all the machine time available, find quantities of x, y and z that can be produced.

Solution  Let \( x \) represent the units of X241 \( y \), units of Youthquake and \( z \) units of Zachangu to be made,

Total time all products use in machine A  \[ 3x + 3y + 2z = 130 \]
For machine B: \[ 3x + 2y = 85 \]
For machine C: \[ x + 3y + z = 60 \]

Numbering the equations we have: \[ 3y + 3x + 2z = 130 \]  (i)
\[ 3x + 2y = 85 \]  (ii)
\[ x + 3y + z = 60 \]  (iii)

In equation (iii) \[ z = 60 - x - 3y \]  (iv)

\[ z \] into eq 1 \[ 3x + 3y + 2(60 - x - 3y) = 130 \]
\[ x - 3y = 10 \]  (v)

Rearranged (v) gives \[ x = 10 + 3y \]  (vi)

(vi) into (ii) \[ 3(10 + 3y) + 2y = 85 \]
\[ 30 + 9y + 2y = 85 \]
\[ 11y = 55 \]
\[ y = 5 \]

\[ y \] into (vi) \[ x = 10 + 3(5) = 10 + 15 = 25 \]
\[ x = 25 \]

Using (iv) \[ z = 60 - x - 3(5) = 60 - 25 - 15 = 20 \]

Therefore \( x = 25, \quad y = 5, \quad z = 20 \).
4.5.3 BUSINESS APPLICATIONS

a) Demand Function

Let \( q \) be the demand (quantity) of a commodity and \( p \) the price of that commodity. The demand function is defined as \( q = f(p) \) where \( p \) and \( q \) are positive. Generally, \( p \) and \( q \) are inversely related.

b) Supply Function

Let \( x \) denote the amount of a particular commodity that sellers offer in the market at various price \( p \), then the supply function is given by \( x = f(p) \) where \( x \) and \( p \) are positive.

c) Cost Function

Normally total cost consists of two parts: (i) Variable cost and (ii) fixed cost. Variable cost is a single-valued function of output, but fixed cost is independent of the level of output. Let \( f(x) \) be the variable cost and \( k \) be the fixed cost when the output is \( x \) units. The total cost function is defined as \( C(x) = f(x) + k \), where \( x \) is positive. Note that \( f(x) \) does not contain a constant term.

Revenue Function

Let \( x \) units be sold at \( p \) Kwacha per unit. Then the total revenue \( R(x) \) is defined as \( R(x) = px \), where \( p \) and \( x \) are positive.

Profit Function

The profit function \( P(x) \) is defined as the difference between the total revenue and the total cost. i.e. \( P(x) = R(x) - C(x) \).

Example 26: If \( R(x) = 240 + 14x \) and \( C(x) = 18x \), find \( P(x) \).

Solution: \( P(x) = R(x) - C(x) \) so \( P(x) = 240 + 14x - 18x = 240 - 4x \).

Example 27: A rental company purchases a truck for K 1,700,000. The truck requires an average of K 1250 per day in maintenance.

a. Find the linear function that expresses the total cost \( C \) of owning the truck after \( t \) days.

b. The truck rents for K 5500 a day. Find the linear function that expresses the revenue \( R \) when the truck has been rented for \( t \) days.

c. Find the profit function \( P(t) \).

d. Use \( P(t) \) to determine how many days it will take the company to break even on the purchase of the truck.

Solution:

a. \( C(t) = 1,700,000 + 1250t \)

b. Here \( R(t) = 5500t \)

c. Since \( P(t) = R(t) - C(t) \) we have

\[
P(t) = 5500t - (1700000 + 1250t) = 4250t - 1700000
\]
d. At break-even point $P(t) = 0$ i.e $4250t - 1700000 = 0$. Solving for $t$ we obtain $t = 400$.

4.5.4 QUADRATIC FUNCTIONS

a) Form of a quadratic function

Quadratic functions are sometimes known as second order functions. The main feature is that the independent variable has power equal to 2.

Generally a quadratic function will take the form:

$$y = ax^2 + bx + c$$

where $a \neq 0$.

Examples 28 The following are typical quadratic equations:

i. $f(x) = x^2 - 5x + 6$
ii. $y = x^2 + 9x + 8$
iii. $g(x)=2x^2 - 11x + 22$
iv. $h(x)=3x^2 + 7x + 4$

b) Solving of a quadratic equation

A quadratic equation can be solved by

i. factorization
ii. formula
iii. graph

Example 29: Solve the following equation:

$$x^2 - 5x + 6 = y$$

Using the three methods

Solution Note: solving a quadratic equation in general means finding the “roots” of the expression. This means finding the value of $x$ when $y = 0$.

Thus we find the value of $y$ when $x = 0$

Or find the value of $x$ in

$$x^2 - 5x + 6 = 0$$

(a) Solution by factorization

$$x^2 - 5x + 6 = 0$$

$$(x - 2)(x - 3) = 0$$

$$x - 2 = 0 \text{ or } x - 3 = 0$$

$$x = 2 \text{ or } x = 3$$

(b) Use of the formula

Given $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
So if \( x^2 - 5x + 6 = 0 \), \( a = 1 \) \( b = -5 \) and \( c = 6 \)

Therefore \( x = \frac{-(-5)\pm\sqrt{(-5)^2-4(1)(6)}}{2(1)} = \frac{5\pm\sqrt{25-24}}{2} = 3 \) or 2.

(c) Graph
To solve a quadratic equation using we need a table of values, plot the points and then read off values of where the curve crosses the x-axis. See the graph below.

Example 30: A company invests in a particular project and it has estimated that after \( x \) months of running, the cumulative profit in thousands of kwacha from the project is given by the expression \( 31.5x - 3x^2 - 60 \) where \( x \) represents time in months, the project can run for a number of months.

(a) Draw a graph to present the profit function.
(b) Calculate the break even time point for the project
(c) What is the initial cost for the project?
(d) Use the graph to estimate the best time to end the project,

Solution a) The graph is shown overleaf.
b) For break even points Profit \( =0 \) i.e. \( y = 31.5x - 3x^2 - 60 = 0 \)

Using the general form
\[
x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}
\]
\[
x = \frac{-31.5\pm\sqrt{(31.5)^2-4(-3)(-60)}}{2(-3)}
\]
\[
x = \frac{31.5\pm\sqrt{992.25-480}}{43}
\]
\[
x = \frac{31.5\pm\sqrt{512.25}}{43}
\]
\[
x = \frac{31.5\pm22.64}{43}
\]
\[
x = 0.84 \text{ or } 5.76
\]
Thus the break even is arrived at in 2.5 months or 8 months

c) The initial cost is at 0 months. This is K60m (equivalent to profit of -K60m

d) The best time to end the project is month 8. Beyond this the profit is negative
4.5.5 Exponential Functions

a) Definition
An exponential function is a function of the form \( f(x) = a^x \) where \( a > 0, \ a \neq 1 \). Here the independent variable occurs as a power of the base \( a \).

Example 31 The following are examples of exponential functions
a) \( f(x) = 2^x \)
b) \( f(x) = \left(\frac{1}{2}\right)^x \)
c) \( f(x) = 4^{-x} \)

We restate the properties of exponents for convenience in terms of variable exponents. We assume that \( a, b > 0 \) and that \( x \) and \( y \) are real numbers. We have:

a) \( a^x a^y = a^{x+y} \)
b) \( (ab)^x = a^x b^x \)
c) \( \frac{a^x}{a^y} = a^{x-y} \)
d) \( \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \)
e) \( (a^m)^x = a^{mx} \)

Example 32 Use the laws of exponents to rewrite and simplify a) \( \frac{4^{-3}}{2^{-8}} \) b) \( 8^{\frac{4}{3}} \)

Solution:

a) \( \frac{4^{-3}}{2^{-8}} = 4^{-3} \div 2^{-8} = (2^2)^{-3} \div 2^{-8} = 2^{-6} \div 2^{-8} = 2^{-6+8} = 2^2 = 4 \)
b) \( 8^{\frac{4}{3}} = (2^3)^{\frac{4}{3}} = 2^{4} = 16 \)

b) The natural exponential function

In mathematics there is a special number which when used as a base for an exponential function yields several useful results. The number is 2.7182818 (to 7 dp) and is usually represented by the letter ‘e’. Thus the natural exponential function is the function \( f(x) = e^x \) and has the same properties as an ordinary exponential function.

4.5.6 Logarithmic Functions

a) Definition
A logarithmic function is a function of the form \( f(x) = \log_a x \) where \( a > 0 \) and \( a \neq 1 \). Exponential functions and logarithmic are closely related. Generally, if \( y = \log_a x \), then \( x = a^y \). This tells us that the logarithm of \( x \) to the base \( a \) is the power to which \( a \) must be raised to obtain \( x \). In the same way if \( x = a^y \), then \( y = \log_a x \). What is the logarithm of 4 to base 2?

Example 33 Without using a calculator, give the exact value of each of the following logarithms.

a) \( \log_2 16 \)
b) \( \log_5 36 \)
c) \( \log_\frac{3}{2} \frac{27}{8} \)

**Solution**
a) \( \log_2 16 = \log_2 2^4 = 4 \).
b) Let \( \log_\frac{3}{5} 36 = y \) then 36 = \( \left( \frac{1}{2} \right)^y \) this can be written as \( 6^2 = (6^{-1})^y = 6^{-y} \). Hence \( y = -2 \) after equating indices.
c) Left to the reader.

**b) Bases of logarithms**

Logarithms can be expressed in different bases but the most commonly used are base 10 and base \( e \). A logarithm with a base 10 is called a common logarithm and if the base is \( e \) it is called the natural logarithm. These logarithms have their own abbreviations:

a) \( \log_{10} x \) is written as \( \log x \)
b) \( \log_e x \) is written as \( \ln x \) read as ‘log of \( x \)’.

Both forms of logarithms can be found on a calculator.

c) **The Operations**

Operations on logarithms follow rules similar to those of ordinary index numbers. The rules of logarithms can be summarized into:

- Product rule
- Quotient rule
- Power rule
- Change of base rule

**Product Rule**
The logarithm of a product is the sum of the logarithms of the factors.

\[
\log_a(xy) = \log_a x + \log_a y
\]

**Example 34** \( \log_{10} (20 \times 5) = \log_{10} 20 + \log_{10} 5 = 1.3010 + 0.6990 = 2 \)

**Quotient Rule**
The logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.

\[
\log_a(x/y) = \log_a x - \log_a y
\]

**Example 35** \( \log_2 \left( \frac{16}{4} \right) = \log_2 16 - \log_2 4 = 4 - 2 = 2 \)

**Power Rule**

\[
\log_a x^n = n \log_a x
\]
Example 36 \[ \log_{10} 16^2 = 2 \log_{10} 16 \]

**Change of Base Rule**

Sometimes, you may be required to convert between bases. Using some simple algebra, a formula can be derived for changing bases:

**Example**

Given a theoretical problem

\[ a^m = b \]

Solve it for \( m \) (i.e. make \( m \) the subject).

**Solution**

There are two ways to solve it

<table>
<thead>
<tr>
<th>Method (a)</th>
<th>Method (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use the log base ( a ): [ \log_a(a^m) = \log_a b ] [ m \log_a a = \log_a b \text{ but } \log_a a = 1 ] So we have ( m = \log_a b )</td>
<td>Take log base ( c ): [ \log_c(a^m) = \log_c b ] [ m \log_c a = \log_c b ] [ m = \log_c b / \log_c a ]</td>
</tr>
</tbody>
</table>

**Example 37**: Find an expression, in terms logarithms to base \( e \), for \( \log_5 10 \) and give an approximate value for the quantity.

**Solution**: From the change-of-base formula, \[ \log_5 10 = \frac{\ln 10}{\ln 5} \approx 1.43 \]

d) **Application of Exponential and Logarithmic Functions**

It is worth noting that exponential and logarithmic functions are inverse operations of one another so a problem involving exponentials may require the use of logarithms. It is for this reason that we treat the applications of exponential and logarithmic functions in one section.

We now consider some of the applications in which exponential and logarithmic functions are used.

e) **Compound Interest**:

If a principal of \( P \) kwacha is invested at an annual rate of interest \( r \), and the interest is compounded \( n \) times per year, then the amount of money \( A(t) \) generated at time \( t \) is given by the formula:

\[ A(t) = P \left(1 + \frac{r}{n}\right)^{nt} \]

**Example 38**: If K600,000 is invested for 3 years at 8% interest compounded annually at the end of each year, what will the final value of the investment be?

**Solution**: Here \( P = K600,000, r = 0.08, t = 3 \) and \( n = 1 \). We now plug these values into the formula above.

\[ A(3) = 600000 \left(1 + \frac{0.08}{1}\right)^{1(3)} = K755,827.20. \]
Example 39: An account earning 10% compound interest has K450,000 as the amount. It is known that the holder invested K180,000 at the beginning, he never added any other deposits and he did not withdraw any money from the account. How long has the account been running to accumulate the stated amount?

Solution: 
\[ A = 257550 \quad P = 180,000, n = 1 \quad \text{and} \quad r = 0.1 \]

The question is about finding \( t \). Let us plug the values into the compound interest formula.

\[ 450,000 = 180000(1 + 0.1)^t = 180000(1.1)^t \]

It is easy to see that the value we are looking for is an exponent. To find \( t \) therefore we need to use logarithms.

We have \( 450,000 = 180000(1.1)^t \) from which we obtain \( \frac{450,000}{180000} = \frac{5}{2} = (1.1)^t \).

Now taking natural logs of both sides we get: \( \ln(1.1)^t = \ln(2.5) \).

Therefore, \( t = \frac{\ln(2.5)}{\ln(1.1)} = 9.61 \) yrs.

CHAPTER SUMMARY

In this chapter we have looked at the following:

- Functions in general (including the definition of a function, domain, range, evaluating and combining functions)
- Types of functions
- Linear functions (general form of linear function, formulating equations from some scenario, the graph of a linear function, using a table of values: the slope of a straight line graph, finding the equation of a line, linear simultaneous equation, solving simultaneous equations system of equations with 3 unknowns and 3 equations and business applications)
- Quadratic functions (including form of a quadratic function solving of a quadratic equation)
- Exponential functions
- Logarithmic functions (definition, product rule, quotient rule, power rule, change of base rule and application of exponential and logarithmic functions)

STUDENT EXERCISES

1. Given \( f(x) = 7x^2 - 3x \) and \( g(x) = -5x^2 - 2x + 6 \) find
   i) \( f(x) + g(x) \)  
   ii) \( f(x) - g(x) \)  
   iii) \( f(x)(g(x)) \)

2. Solve the equation: \( 4x^2 + 3x - 2.5 = 0 \).

3. True or False? If \( R(x) = 124x, \) and \( C(x) = 78-5x + 5005 \) then the profit function is \( P(x) = 45.5x + 5005 \).

4. The length of a rectangle is 1cm more than twice its width. Represent the width of the rectangle by \( w \) and write a function to express the perimeter of the rectangle in terms of \( w \).
5. The functions \( R(x) = 100x \) and \( C(x) = 2000x + 50x \) represent revenue and costs of an organisation. Write out the profit function.

6. Given that the price of an item is K3.50 when 250 items are demanded, but when only 50 are demanded the price rises to K5.50 per item, identify the linear demand function and calculate the price per item at a demand level of 115.

7. Assume that the supply function is represented by \( P = 0.05q + 10 \), and the demand function is represented by \( P = 17 - 0.02q \), where \( P \) is the price in K'000 and \( q \) is the quantity.

**Required:**
If equilibrium price is defined as a situation where demand equals supply, find the equilibrium price and quantity.

8. The supply of a commodity is related to the price by the relation \( P = 5\sqrt{2p} - 10 \). Show that the supply curve is a parabola. Find its vertex and the price below which supply is 0?

9. An air freight company has determined that its cost of delivering \( x \) parcels per flight is \( C(x) = 2025 + 7x \). The price it charges to send \( x \) parcels is \( p(x) = 22 - 0.01x \).

Determine  
(a) The revenue function.  
(b) The profit function.  
(c) The company's maximum profit.  
(d) The price per parcel that yields the maximum profit.  
(e) The minimum number of parcels the air freight company must ship to break even.
CHAPTER 5   SEQUENCES AND SERIES

OBJECTIVES

By the end of this chapter students should be able to:

i. Define a sequence and series
ii. Give examples of a sequence
iii. Give examples of a series
iv. Define an Arithmetic Progression
v. Identify an Arithmetic Progression
vi. Identify a geometric progression
vii. Determine the $n^{th}$ terms of arithmetic progressions and geometric progressions
viii. Find the sum of a given AP
ix. Find the sum of a given GP

5.0 INTRODUCTION

In general mathematics deals with patterns, whether they are visual patterns or numerical patterns. For example, exponential growth is a growth pattern that is shared by populations, bank accounts etc. Sequences and Series deal with numerical patterns. In this chapter we will look at two types of sequences- arithmetic and geometric sequences, but first we start with a general overview of sequences and series.

5.1 SEQUENCES AND SERIES OVERVIEW

5.1.1 Sequences

A sequence is a list of numbers, called terms arranged in a definite order.

Example 1

i. 3, 7, 11, 15, ...

In this sequence each term is obtained by adding 4 to the previous term. So the next term would be 19.

ii. 4, 9, 16, 25, ...

This sequence can be rewritten as $2^2, 3^2, 4^2, 5^2, ...$. The next term would be $6^2$ or 36.

The dots (...) indicate that the sequence continues indefinitely – an infinite sequence.

A sequence such as 3, 6, 9, 12 (stopping after a finite number of terms) is a finite sequence.

Suppose we write $a_1$ for the first term of a sequence, $a_2$ for the second and so on. There maybe a formula for the $n^{th}$ term $a_n$:  

50
Example 2
The $n^{th}$ term of a sequence

(i) $4, 9, 16, 25, \ldots$ The formula for the $n^{th}$ term is $a_n = (n + 1)^2$.

(ii) The sequence : $5, 7, 9, 11, \ldots$, would be given by the formula: $a_n = 2n + 3$

(iii) $a_n = 2n + 1$. The sequence is: $3, 5, 7, 9, \ldots$

5.1.2 Series
A series is formed when the terms of a sequence are added together. The Greek letter $\sum$ (pronounced “sigma”) is used to denote “the sum of”:

$$\sum_{i=1}^{n} a_i \text{ means } a_1 + a_2 + \cdots + a_n$$

Examples 3
(i) In the sequence $3, 6, 9, 12, \ldots$, the sum of the first five terms is the series is

$$3 + 6 + 9 + 12 + 15$$

(ii) $\sum_{i=1}^{6} (2i + 3) = 5 + 7 + 9 + 11 + 13 + 15$

5.2 ARITHMETIC PROGRESSION (AP)
An arithmetic progression is a sequence in which each term, after the first, can be obtained by adding a fixed number, called the common difference, to the previous term.

Example 4
The two sequences below are both arithmetic progressions

i. $1, 4, 7, 10, \ldots$ Common difference is $3$.

ii. $13, 7, 1, \ldots$ Common difference is $-6$

5.2.1 $n^{th}$ term of an arithmetic progression
Suppose an AP has the first term $a$ and a common difference $d$ then the AP will take the form

$$a, a + d, a + 2d, a + 3d, \ldots$$

$$= a, \quad a + (2 - 1)d, \quad a + (3 - 1), \quad a + (4 - 1)d, \quad \ldots$$

From above we can therefore deduce that given the first term $a$ and common difference $d$ of an AP, the term on any position $n$ of the progression would be given by

$$a_n = a + (n - 1)d$$
Where \( a_n \) is the \( n^{th} \) term of the AP.

**Example 5**  Find the \( 6^{th} \) term of the AP: 1, 4, 7, 10, ...

**Solution**

\[
\begin{align*}
n^{th} \text{ term} &= a + (n - 1)d \\
a &= 1; d = 3
\end{align*}
\]

Hence

\[
6^{th} \text{ term} = 1 + (6 - 1)3 = 16
\]

**Example 6** Find the \( 14^{th} \) term of the AP 20, 17, 14, 11, 8, ...

**Solution**

\[
\begin{align*}
a &= 20; d = -3 \\
14^{th} \text{ term} &= 20 + (14 - 1) \times (-3) \\
&= -19
\end{align*}
\]

**Example 7** The Salary of a teacher grows by K3,000 each year. If the starting salary in his grade is K18,000, find the teacher’s salary in the fifth year.

**Solution**

The teacher’s salary at the start of the start of each year would be

\[K18,000, K18,000 + K3,000, K18,000 + K6,000, ...\]

which is an AP with a common difference \( K3,000 \)

Hence the salary at the start of the fifth year would be:

\[K18,000 + (5 - 1) \times K3,000 = K30,000\]

### 5.2.2 Sum of an Arithmetic Progression

For an AP with a first term \( a \) and common difference \( d \) the sum of the first \( n \) terms of the progression is given by

\[
S_n = \frac{n}{2} [2a + (n - 1)d]
\]

Where \( S_n \) is the sum of the first \( n \) terms of the progression.

**Example 8** Find the sum of the following sequence.

2, 5, 8, 11, 14, 17.
Solution

Adding each of the terms we get the sum

\[ 2 + 5 + 8 + 11 + 14 + 17 = 57 \]

However, we note that the above sequence is an AP of 6 terms with the first term 2 and common difference 3

Using the formula

\[ S_n = \frac{n}{2} [2a + (n - 1)d] \]

To find the sum of those six terms of the AP we have

\[ S_6 = \frac{6}{2} [2 \times 2 + (6 - 1) \times 3] \]
\[ = 3 \times (4 + 5 \times 3) \]
\[ = 57 \]

Example 9 A firm rents out its premises and the rental agreement provides for a regular annual increase at K10,000. If the rent in the first year is K85,000.00

a) What is the rent in the 10\textsuperscript{th} year?

b) How much will the firm have paid cumulatively in the 10 years.

Solution

a) 10\textsuperscript{th} year rent is the 10\textsuperscript{th} term of the sequence of annual payments which form an AP with first term K85,000 and common difference K10,000

\[ 10\textsuperscript{th} term = K85,000 + (10 - 1) \times K10,000 \]
\[ = K175,000 \]

b) Cumulative rental is the sum of the first 10 rentals.

\[ S_n = \frac{n}{2} [2a + (n - 1)d] \]
\[ S_{10} = \frac{10}{2} [K85,000 \times 2 + (10 - 1) \times 10,000] \]
\[ = 5(K170,000 + K90,000) \]
\[ = K1,300,000 \]
5.3 GEOMETRIC PROGRESSION (GP)

A geometric Progression is sequence in which each term can be obtained by multiplying the previous term by a fixed number, called the common ratio.

Examples of Geometric Progression

i. 1, 2, 4, 8, ...
   Each term is double the previous one. The common ratio is 2.

ii. 81, 27, 9, 3, 1, ... The common ratio is $\frac{1}{3}$

5.3.1 The $n^{th}$ term of a geometric progression

Suppose a geometric progression has a first term $a$ and a common ratio $r$, then the progression will take the form

$$a, ar, ar^2, ar^3, ...$$

$$= a, ar^{2-1}, ar^{3-1}, ar^{4-1}, ...$$

It can therefore be deduced from above that the $n^{th}$ term of the progression at any position $n > 1$ will be given by

$$a_n = ar^{n-1}$$

Example 10

a) Find the sixth term of the sequence 1, 2, 4, 8, ...

b) The first term in a GP is 1000 and its common ratio is 0.8. Find the 5th term.

Solution

We note that each term is twice its predecessor hence the sequence is a geometric progression with the first term 1 and common ratio 2.

a) $n^{th}$ term: $a_n = ar^{n-1}$

Hence the 6th term of the progression is

$$a_6 = 1 \times 2^{6-1}$$
$$= 1 \times 2^5$$
$$= 32$$
Solution

\[ n^{th} \text{ term: } a_n = ar^{n-1} \]

\[ 5^{th} \text{ term } = 100 \times 0.8^{5-1} \]

\[ = 40.96 \]

5.3.2 Sum of a Geometric Progression

For a geometric progression with a first term \( a \) and a common ratio \( r \) the sum of the first \( n \) terms of the progression is given by

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]

\[ = \frac{a(1 - r^n)}{1 - r} \]

If the common ratio \( r \) has a numerical value lying between \(-1\) and \(1\) that is \(|r| < 1\), then the sum of an infinite number of terms of the series denoted \( S_\infty \) or \( S_{n \to \infty} \) is given by

\[ S_{n \to \infty} = \frac{a}{1 - r} \]

Example 11

Find the sum of the first 5 terms in the GP 2, 4, 8,...

Solution

Sum of a geometric progression:

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]

\[ a = 2; r = 2; n = 5 \]

\[ S_n = \frac{2(2^5 - 1)}{2 - 1} \]

\[ = \frac{2(32 - 1)}{1} \]

\[ = 62 \]

Example 12

Find the sum of the infinite sequence 1000, 500, 250, 125, ...

Solution

Note that each term in the sequence after the first can be obtained by multiplying its predecessor by \( \frac{1}{2} \)
Hence $a = 1000; r = 0.5, n \to \infty$

And

$$S_{n\to\infty} = \frac{a}{1-r}$$

$$= \frac{1000}{1-0.5}$$

$$= 2000$$

**CHAPTER SUMMARY**

In this chapter, we have looked at sequences and series with a special focus on

- Arithmetic Progression
  - Definition of an Arithmetic Progression
  - Finding the $n^{th}$ term of an Arithmetic Progression
  - Sum of an Arithmetic Progression
- Geometric Progression
  - Definition of a Geometric Progression
  - Finding the $n^{th}$ term of a Geometric Progression
  - Sum of a Geometric Progression

**END OF CHAPTER EXERCISES**

1. Insert 5 numbers between 8 and 26 such that the resulting series is an AP.

2. The fifth term of an arithmetic progression is 26 and twelfth is 75. Find the eighth term.

3. Find the first term of an arithmetic progression given that the $20^{th}$ term = 100 and the $22^{nd}$ term = 108.

4. An employee, who received fixed annual increments had a final salary of K90,000 after 10 years. If her total salary was K650,000 over the 10 years what was her initial salary?

5. James has just been offered a new job. His initial salary is K1, 500,000 per annum, with an annual increment of K50,000. If this does not change how much will he earn in total over a period of 12 years?

6. Which term in the GP 2, 4, 8, 16 is 1024

7. Find the sixth term of the geometric progression
   a. 128, 96, 72, ...

56
b. 3, -6, 12, . . .

8. The third term of a geometric progression is 6, and the eighth term is 192. Find the first term of the progression and the common ratio,

9. A business man saves from his profits as follows: K3,000 in the first week, K6,000 in the second week, K12,000 in the third week, K24,000 in the fourth week, and so on. If he keeps on with this pattern of savings,

a. How much would he save in the tenth week?

b. How much would be his total savings in the first ten weeks?
CHAPTER 6 INEQUALITIES

LEARNING OBJECTIVES

By the end of this chapter the student should be able to:

i. Define an inequality
ii. Formulate simple linear inequalities
iii. Solve linear inequalities
iv. Sketch a feasible region given linear constraints
v. Apply inequalities to simple commercial situations

6.0 INTRODUCTION

The concept of inequalities is similar to that of equations. The difference is that in an equation one side is identical in value to the other. In inequalities one side is not necessarily identical to the other. It may be ‘greater than’, ‘greater than or equal to’, ‘less than’, or ‘less than or equal to’.

Just like with equations, the solution to an inequality is a value that makes the inequality true. You can solve inequalities in the same way you solve equations, by following the rules.

Inequalities may be linear or non-linear expressions.

6.1 SYMBOLS USED

< means “less than”
≤ means “less than or equal to”
> means “greater than”
≥ means “greater than or equal to”

Example 1 The following expressions are all inequalities

a. $3x + 4 > 2$
b. $x - 5 \geq 0$
c. $2 - 3x < 8$
d. $x + 3 \leq 5$
e. $x^2 + x - 6 > 0$

Inequalities a) to d) are all linear while e) is non-linear

6.2 SOLVING LINEAR INEQUALITIES.
6.2.1 Addition and subtraction of a number

Any positive or negative number may be added to both sides of an inequality.

Example 2

Solve

a. \( x - 5 \geq 0 \)

b. \( x + 3 \leq 5 \)

Solution:

a. Add 5 to both sides to find \( x \):

\[
x - 5 + 5 \geq 0 + 5
\]

\[
x \geq 5
\]

b. Subtract 3 from both sides to find \( x \):

\[
x + 3 - 3 \leq 5 - 3
\]

\[
x \leq 2
\]

6.2.2 Multiplication or division of an inequality by a number

An inequality can be multiplied or divided by a positive number on both sides in the process of simplifying the expression without affecting the direction of the inequality symbol. However, if an inequality is multiplied or divided by a negative number, the direction of the inequality sign is reversed.

Example 3

Solve

a. \( \frac{x}{3} < 2 \)

b. \( 6x > 18 \)

c. \( -\frac{x}{6} > 4 \)

d. \( -4x > 24 \)

Solution

a. \( \frac{x}{3} < 2 \), We multiply both sides by 3 to get \( 3 \times \frac{x}{3} < 2 \times 3 \Rightarrow x < 6 \).

b. \( 6x > 18 \). Here we divide both sides by 6, that is, \( \frac{6x}{6} > \frac{18}{6} \Rightarrow x > 3 \).

c. \( -\frac{x}{6} > 4 \). To make \( x \) the subject we have to multiply both sides by \(-6\), doing that gives \( -6 \times -\frac{x}{6} < 4 \times -6 \Rightarrow x < -24 \).

d. \( -4x > 24 \). Here we will divide both sides by -4. Let us do that, \( \frac{-4x}{-4} < \frac{24}{-4} \Rightarrow x < -6 \).

6.2.3 Taking the reciprocal of an inequality

Given an inequality, the direction of the sign changes if both sides are inverted.

59
Example 4  Take the reciprocal of the following inequality:

\[ x > 5 \]

Solution  \( \frac{1}{x} < \frac{1}{5} \)

Key points

i)  Adding or subtracting any number does not change the direction of the sign.

ii) Multiplying or dividing by a positive sign does not change the direction of the sign.

iii) Multiplying or dividing by a negative number reverses the inequality symbol.

iv) The reciprocal of the inequality reverses the sign.

6.2.4. Solving General Inequalities

As pointed out earlier, inequalities are solved the same way we solve equations but we have to remember that when we multiply or divide an inequality by a negative number or take the reciprocal of an inequality, we have to change the direction of the inequality sign.

Example 5  Solve the following inequalities

a) \( 4x - 3 > 10 \)

Solution  \( 4x > 13 \)
\( x > 3.25 \)

b) \( 5 - 3\frac{1}{2}y < 21 \)

Solution  \( -3\frac{1}{2}y < 21 \)
\( -\frac{7}{2}y < 21 \)
\( y > -6 \) (Note the change in the sign)

6.2.5 GRAPH OF INEQUALITIES

While the graph of \( y = mx + c \) is a line passing through all points which satisfy \( y = mx + c \), the graph of an inequality, for example \( y < mx + c \) defines a region containing solutions to the inequality.

To draw the graph of a linear inequality:

Step 1: Proceed the same way we graph a linear function.
Step 2: Draw a dotted line if the inequality is strict (< or >). Else draw a solid line.
Step 3: Now choose a test point.
Step 4: Plug the test point into the given inequality to test if it satisfies the given inequality.
Step 5: If it does, shade the opposite region ELSE shade the region containing the test point.

Example 8  Draw the graph of \( x > 2 \)

Solution  First draw the graph of \( x = 2 \). This is a vertical line through \( x = 2 \)
The un-shaded area to the right of the graph contains solutions to $x>2$.

You will have noticed in the graph of the previous inequality that we shaded the unwanted region but other authors prefer to shade the wanted region. As such do not be surprised if you come across a book in which the region containing solutions to the inequality is shaded. In an exam, where all working is done by hand, we advise that you shade the unwanted side as doing that leaves the region containing solutions to the inequality clear and not messy.

Let us look at an example where the wanted side is shaded.

Example 9 Draw the graph of $x>y$.

Solution:

Step 1: Draw the graph of $x = y$. This will be a dotted line through the origin.
Step 3: Now choose $(2,0)$ as a test point.
Step 4: $(2,0)$ means $x = 2$ and $y = 0$ so the inequality becomes $2 > 0$ which is true. So the side containing $(2,0)$ is the wanted side.
Step 5: Since in this case we are shading the wanted side, we will shade the region containing the point $(2,0)$ as shown below.
Example 10. Draw the graph of $y < 2x + 2$.

**Solution:**

Step 1: Draw the graph of $y = 2x + 2$. This will be a solid line through $(0,2)$ and $(-1,0)$. Here we are just plotting the intercepts.

Step 3: Now choose $(1,0)$ as a test point.

Step 4: $(1,0)$ means $x = 1$ and $y = 0$ so the inequality becomes $0 < 3$ which is true. So the side containing $(1,0)$ is the wanted side.

Step 5: Since in this case we are shading the unwanted side, we will shade the region not containing the point $(1,0)$ as shown below.

The following is the graph:
FIGURE 6.3 Graph of \( y < 2x + 2 \)

CHAPTER SUMMARY

In this chapter we have looked at the following:

- Symbols used when describing an inequality.
- Solving inequalities.
- Graphing linear inequalities

END OF CHAPTER EXERCISES

1. Solve the following inequalities for \( x \)
   
   a) \( x + 5 > 3 \)  
   b) \( 5 - 3x \leq 14 \)  
   c) \( 9x + 9 \geq 2x - 19 \)  
   d) \( 3(2x + 7) - 5 \leq 4(x + 1) - x \)  
   e) \( 10x - 5 < 45 \)

2. Solve \( \frac{3x + 2}{6} - \frac{1}{3} \leq x \)

b) \( 15/4 > 1/ \)
3 Zione the Sells The Daily times by direct delivery to homes in Area 47, Lilongwe. For each Newspaper delivered she earns K5 and The Newspaper company gives her K5000 each regardless of how many Newspapers she is selling. If Zione wants to earn at least K40000 this week, what is the minimum number of subscriptions she needs to sell?

4 Wezi a self boarding Accountancy student and studying in Blantyre has K30,000 in her account at the start of the semester which has 21 weeks. She would like to save at K5000 for her transport back home in Lilongwe. Write out an inequality to express how much Wezi should withdraw from the account per week for her upkeep.
CHAPTER 7  LINEAR PROGRAMMING

OBJECTIVES

By the end of this chapter the students should be able to:

i. Find the objective function for a given problem.
ii. Come up with constraints from a given word problem.
iii. Plot inequalities and determine a feasible region for a given word problem.
iv. Find the optimal value of a function for a given word problem.

7.0  INTRODUCTION

This chapter seeks to introduce you to linear programming which is concerned with the problem of optimizing (maximizing or minimizing) some variable (profit, output, loss etc) subject to some constraints (costs, availability etc). Linear programming therefore enables a manager to calculate the profit-maximizing output mix of a multi-product firm subject to restrictions on input availability, or the input mix that will minimize costs subject to minimum quality standards being met. As such linear programming is an extremely useful tool for managerial decision-making.

7.1  DEFINITION OF TERMS

In the solution of a linear programming problem one usually meets the terms defined below:

Objective Function: This is the function to be maximised or minimised.

Constraints: Linear equations or inequalities which restrict the values of the variables.

Feasible Region: A region in the x-y plane which satisfies all of the constraints under consideration.

Let us consider the case of a manager who wishes to decide on the product mix which will maximize profits when his firm has limited amounts of the various inputs required for the different products that it makes. The firm’s objective is to maximize profit and so profit is the objective function. He will try to optimize this function subject to the constraint of limited input availability. The constraints (there will be many) determine a feasible region when plotted.

7.2  LINEAR PROGRAMMING MODEL

A general linear programming model takes the following form:

To do: Optimise \( ax + by \).

Subject to:

\[
\begin{align*}
    a_1 x + b_1 y &< c_1 \\
    a_2 x + b_2 y &< c_2 \\
    \vdots \\
    a_n x + b_n y &< c_n
\end{align*}
\]

where \(<\) can also be \(\leq, \geq\) or \(=\).

This tells us that the optimal value of the objective function \( P = ax + by \) will be found subject to a given number of constraints (\( n \) in this case).

There are various ways of solving a linear programming problem, the only one relevant to this course is the graphical approach.
7.3 GRAPHICAL APPROACH

We give the procedure for solving a linear programming problem.

Input: Objective function \( P = ax + by \) and a set of constraints.

Output: Optimal value of the objective function.

Step 1: Plot the lines corresponding to each of the given constraints.

Step 2: Determine the feasible region. This will be the region satisfying all the given inequalities.

Step 3: The maximum or minimum value of the objective function occurs at a vertex of the feasible region. So to work out the optimum solution, substitute values of \( x \) and \( y \) from each vertex of the feasible region into the given objective function. Now determine the maximum value or minimum value by considering values of the objective function. The largest value of the objective function corresponds to a vertex which maximizes the objective and the smallest value of the objective function corresponds to a vertex which minimizes the objective function.

Note: It is not always the case that the optimal solution lies, as some students would like to believe, where the constraints they have drawn intersect. See problem 6 in the exercises.

Example 1: Find the maximum and minimum values of \( 2x - y \) subject to the constraints \( 2x + y \leq 10, x + y \leq 6 \) and \( x \geq 0, y \geq 0 \).

Solution:

Here we would like to optimize \( P = 2x - y \)

Subject to the constraints \( 2x + y \leq 10, x + y \leq 6 \) and \( x \geq 0, y \geq 0 \).

We will plot the inequalities to obtain the feasible and then plug vertices of the feasible region into \( P = 2x - y \). The diagram below shows the feasible region.
The feasible region has four vertices 

\((0,0), (0,6), (5,0)\) and \((4,2)\). Now to optimize the objective function \(P = 2x - y\) we will plug each vertex into \(P = 2x - y\). Let us do that.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>(P = 2x - y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0)) i.e (x = 0) and (y = 0)</td>
<td>(P = 2(0) - 0 = 0)</td>
</tr>
<tr>
<td>((0,6)) i.e (x = 0) and (y = 6)</td>
<td>(P = 2(0) - 6 = -6)</td>
</tr>
<tr>
<td>((4,2)) i.e (x = 4) and (y = 2)</td>
<td>(P = 2(4) - 2 = 6)</td>
</tr>
<tr>
<td>((5,0)) i.e (x = 5) and (y = 0)</td>
<td>(P = 2(5) - 0 = 10)</td>
</tr>
</tbody>
</table>

We can see that the maximum value is 10 and minimum value -6.

Example 2: A company manufactures two types of boxes, corrugated and ordinary cartons. The boxes undergo two major processes: cutting and pinning operations. The profits per unit are \(K\ 6\) and \(K\ 4\) respectively. Each corrugated box requires 2 minutes for cutting and 2 minutes for pinning operation, whereas each carton box requires 3 minutes for cutting and 1 minute for pinning. The available operating time is 120 minutes and 60 minutes for cutting and pinning machines respectively. Determine the optimum quantities of the two boxes to maximize the profits.

Solution:

**Key Decision:** To determine how many (number of) corrugated and carton boxes are to be manufactured.
TABLE: We present the information given in a table. You will notice that if we use a table coming up with the constraints is straightforward.

<table>
<thead>
<tr>
<th>Process</th>
<th>Corrugated</th>
<th>Ordinary</th>
<th>Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting</td>
<td>2</td>
<td>3</td>
<td>120</td>
</tr>
<tr>
<td>Pinning</td>
<td>2</td>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td>Profit</td>
<td>K6</td>
<td>K4</td>
<td></td>
</tr>
</tbody>
</table>

Decision variables: Let \( x \) and \( y \) be the number of corrugated and carton boxes to be manufactured respectively.

Objective Function: The objective is to maximize the profits. Given profits on corrugated box and carton box are K 6 and K 4 respectively. Therefore the objective function is, \( P = 6x + 4y \).

Constraints: The available machine-hours for each machine and the time consumed by each product are given. Therefore, the constraints are,

\[
2x + 3y \leq 120 \\
2x + y \leq 60
\]

As the company cannot produce negative quantities of the two goods, we can also add the two non-negativity constraints on the solutions for the optimum values \( x \geq 0 \) and \( y \geq 0 \). We now plot the inequalities to determine the feasible region. See the graph below.

![Figure 7.2 Inequalities and Feasible region](image)
Note that the coordinates for B can be read off from the graph or can be found by solving the equations \(2x + 3y = 120\) and \(2x + y = 60\).

The feasible region is clear from the graph and contains the vertices \((0,0)\), \((0,40)\), \((15,30)\) and \((30,0)\). The next thing is to plug these into the objective function. We will do this using a table.

**Table 7.2 Feasible region vertices and profit levels**

<table>
<thead>
<tr>
<th>Vertex</th>
<th>(P = 6x + 4y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0)) i.e (x = 0) and (y = 0)</td>
<td>(P = 6(0) + 4(0) = 0)</td>
</tr>
<tr>
<td>((0,40)) i.e (x = 0) and (y = 40)</td>
<td>(P = 6(0) + 4(40) = 160)</td>
</tr>
<tr>
<td>((15,30)) i.e (x = 15) and (y = 30)</td>
<td>(P = 6(15) + 4(30) = 210)</td>
</tr>
<tr>
<td>((30,0)) i.e (x = 30) and (y = 0)</td>
<td>(P = 6(30) + 4(0) = 180)</td>
</tr>
</tbody>
</table>

It is not difficult to see that 15 corrugated boxes and 30 ordinary boxes will yield the maximum profit.

**CHAPTER SUMMARY**

In this chapter we have looked at the following:

- Definitions of Constraint, Objective function and Feasible Region.
- Graphing a system of linear inequalities to obtain a feasible region.
- Solving linear programming problems using the graphical method.

**END OF CHAPTER EXERCISES**

1. Find the values \(x\) and \(y\) that maximize the function \(P = 2x + 4y\) subject to the constraints: \(x \leq 400, y \leq 300, x + y \leq 600\) and \(x \geq 0, y \geq 0\).

2. Show that more than one solution exists if one tries to maximize the objective function \(P = 4x + 4y\) subject to the constraints:
   \[
   \begin{align*}
   20x + 20y & \leq 60 \\
   20x + 80y & \leq 120 \\
   x & \geq 0 \quad By \geq 0
   \end{align*}
   \]

3. Minimize the objective function \(C = 12x + 8y\) subject to the constraints
   \[
   \begin{align*}
   10x + 40y & \geq 40 \\
   12x + 16y & \leq 48 \\
   x & = 1.5
   \end{align*}
   \]

4. Find the minimum value of the function \(C = 40A + 20B\) subject to the constraints
   \[
   \begin{align*}
   10A + 40B & \geq 40 \\
   30A + 20B & \geq 60
   \end{align*}
   \]
5. Wezi a self boarding Accountancy student and studying in Blantyre has K30,000 in her account at the start of the semester which has 21 weeks. She would like to save at K5000 for her transport back home in Lilongwe. Write out an inequality to express how much Wezi should withdraw from the account per week for her upkeep.

6. A company uses inputs K and L to manufacture goods A and B. It has available 200 units of K and 180 units of L and the input requirements are:
   - 10 units of K plus 30 units of L for each unit of A
   - 25 units of K plus 15 units of L for each unit of B
If the per-unit profit is K80 for A and K30 for B, what combination of A and B should it produce to maximize profit and how much of K and L will be used in doing this?

7. Mr Chiwoza is a farmer in Chitipa, a wheat and paprika growing area. He has 10 acres to plant in wheat and paprika. Mr. Chiwoza has to plant at least 7 acres. However, he has only K360,000 to spend and each acre of wheat costs K60,000 to plant and each acre of paprika costs K30,000 to plant. Moreover, the farmer has to get the planting done in 12 hours and it takes an hour to plant an acre of wheat and 2 hours to plant an acre of paprika.

Required

a) Identify the constraints for Mr Chiwoza and write them as inequalities.
b) Graph the inequalities and identify the feasible region.
CHAPTER 8  MATRICES

LEARNING OBJECTIVES

By the end of this chapter the student should be able to:

i. Define a matrix
ii. Distinguish types and orders of matrices
iii. Understand the equality of matrices
iv. Represent data with matrices
v. Add, subtract and multiply matrices
vi. Apply the concept of matrices in manipulating commercial data
vii. Find the determinant of matrices up to 3 by 3.
viii. Find the inverse of 2 × 2 and 3 by 3 matrices.
ix. Solve systems of linear equations (up to 3 variables) using the inverse method and Cramer’s rule.

8.0 INTRODUCTION

You might have seen that there are so many ways in which numerical information is given or stored. One of such ways is by using tables. Below is a table showing student grades in Mathematics, Communication, Accounts and Computing for 3 best students at Payelepayele Business School

Table 8.1: Semester I results

<table>
<thead>
<tr>
<th>Name</th>
<th>Mathematics</th>
<th>Communication</th>
<th>Accounts</th>
<th>Computing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chimwemwe</td>
<td>80</td>
<td>78</td>
<td>55</td>
<td>68</td>
</tr>
<tr>
<td>Atupele</td>
<td>82</td>
<td>70</td>
<td>50</td>
<td>65</td>
</tr>
<tr>
<td>Ayali</td>
<td>56</td>
<td>80</td>
<td>70</td>
<td>50</td>
</tr>
</tbody>
</table>

Note that in the above table, the grades have been arranged in 3 rows and four columns.

In the second semester, the three pupils obtained the following results:

Table 8.2: Semester II results

<table>
<thead>
<tr>
<th>Name</th>
<th>Mathematics</th>
<th>Communication</th>
<th>Accounts</th>
<th>Computing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chimwemwe</td>
<td>76</td>
<td>70</td>
<td>65</td>
<td>72</td>
</tr>
<tr>
<td>Atupele</td>
<td>70</td>
<td>80</td>
<td>60</td>
<td>55</td>
</tr>
<tr>
<td>Ayali</td>
<td>68</td>
<td>70</td>
<td>70</td>
<td>60</td>
</tr>
</tbody>
</table>

We will now look at ways in which we can analyse this information given in the tables.

Example 1
A class tutor is interested in finding the average grades of the students. Show the final grades of the three students in a table.
Solution

It is easy to see that to get a grade of a pupil in a particular subject, we will add the two grades of a pupil corresponding to each subject and then divide by 2.

Table 8.3: Average grades

<table>
<thead>
<tr>
<th>Name</th>
<th>Mathematics</th>
<th>Communication</th>
<th>Accounts</th>
<th>Computing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chimwemwe</td>
<td>80 + 76</td>
<td>76 + 70</td>
<td>55 + 65</td>
<td>68 + 72</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Atupele</td>
<td>82 + 70</td>
<td>70 + 80</td>
<td>50 + 60</td>
<td>65 + 55</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Ayali</td>
<td>56 + 68</td>
<td>80 + 70</td>
<td>70 + 70</td>
<td>50 + 60</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Simplifying, we get:

Table 8.4

<table>
<thead>
<tr>
<th>Name</th>
<th>Mathematics</th>
<th>Communication</th>
<th>Accounts</th>
<th>Computing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chimwemwe</td>
<td>78</td>
<td>73</td>
<td>60</td>
<td>70</td>
</tr>
<tr>
<td>Atupele</td>
<td>76</td>
<td>75</td>
<td>55</td>
<td>60</td>
</tr>
<tr>
<td>Ayali</td>
<td>62</td>
<td>75</td>
<td>70</td>
<td>55</td>
</tr>
</tbody>
</table>

Note again that by adding and dividing by 2 the corresponding entries of the two tables with 3 rows and 4 columns, we have ended up with a table having the same 3 rows and 4 columns.

The method of keeping information using tables can be simplified by simply presenting the numbers in rows and columns rounded with brackets.

Thus the information, say from Table 8.4, could simply appear as shown below

\[
\begin{pmatrix}
78 & 73 & 60 & 70 \\
76 & 75 & 55 & 60 \\
62 & 75 & 70 & 55 \\
\end{pmatrix}
\]

Thus we have conveniently represented Table 8.4 as a rectangular array of numbers, a mathematical word of describing this is MATRIX and to show that the matrix has 3 rows and 4 columns, we say that it is a 3 by 4 matrix. The following definitions summarise the ideas.

**8.1 DEFINITION OF KEY TERMS**

A matrix (plural matrices) is a rectangular array of numbers arranged by rows and columns. These numbers are the elements of the matrix and usually a matrix is written by enclosing the elements in
brackets. It is also usual to name a matrix using a letter. A matrix $M$ with $r$ rows and $c$ columns is said to be an $(or a) r \times c$ which is read as ‘$r$ by $c$’.

The following are illustrations of matrices.

**Example 2**

We look at the following matrices

a) $A = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 \\ 6 & -1 & 0 & 2 \end{pmatrix}$, b) $B = \begin{pmatrix} -2 & 0 & 3 & 4 \end{pmatrix}$

c) $C = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 3 \\ 5 \end{pmatrix}$  
d) $X = \begin{pmatrix} 3 & 9 \\ 8 & 7 \end{pmatrix}$

Thus since matrix $A$ has 3 rows and 4 columns, it is a $3 \times 4$ matrix. While $B$ has one row and 4 columns hence it is a $1 \times 4$ matrix and such matrices are referred to as *row matrices*. $C$ has 5 rows and 1 column and so it is a $5 \times 1$ matrix. By virtue of having one column, it is referred to as a *column matrix*. Lastly, $X$ is a $2 \times 2$ matrix and because the number of rows is equal to the number of columns, such matrices are referred to as *square matrices*.

### 8.2 IDENTIFICATION OF ELEMENTS

Recall that a matrix is a rectangular array of numbers arranged in rows and columns. This means each element will be in a particular row and column. The rows and columns make a grid which can be used to uniquely identify an element.

The practice is to start with row then column.

**Example 3**

In matrix $X = \begin{pmatrix} 3 & 9 \\ 8 & 7 \end{pmatrix}$, 3 can be identified as the element in row 1 column 1 and 8 is in row 2 column 1 and so on.

Using this way of identifying elements, a matrix can be written in general as follows

Matrix $X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and, in general, $a_{i,j}$ represents an element in the $i^{th}$ row and $j^{th}$ column.
8.3 ORDER OF MATRICES
The term “order” (or dimension) in the context of matrices refers to the size of the matrix and it is measured in terms of the number of rows and columns.

In section 8.1 above, A above is a $3 \times 4$ (3 by 4) matrix, B is a $1 \times 4$ matrix, C is a $5 \times 1$ matrix while X is a $2 \times 2$ matrix.

Example 4

State the order of the following matrices and, where possible, what is the value of element $a_{23}$ in each of the matrices?

(a) $A = \begin{pmatrix} -2 & 0 & -1 & 2 & 1 \\ 3 & 5 & -2 & 3 & 0 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & 0 \\ 4 & 1 & -2 \end{pmatrix}$

c) $C = \begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \end{pmatrix}$

d) $D = \begin{pmatrix} 4 & 3 & 0 & -1 \end{pmatrix}$

Solution

a) A is a $2 \times 5$ matrix while $a_{23} = -2$

b) B is a $3 \times 3$ matrix while $a_{23} = 0$

c) C is a $4 \times 1$ matrix while $a_{23}$ does not exist since there is no $3^{rd}$ column.

d) D is a $1 \times 4$ matrix while $a_{23}$ does not exist since there is no $2^{nd}$ row.

8.4 IMPORTANT TYPES OF MATRICES

8.4.1 The square matrix
A square matrix is a matrix which has its number of rows equal to the number of columns.

Example 5

The following are all square matrices

(a) $M = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$

(b) $Q = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & 0 \\ 4 & 1 & -2 \end{pmatrix}$
8.4.2 The identity matrix

This is a square matrix whose elements consist of ones in the leading diagonal and zeros for the other elements. It is usually denoted by letter I. The identity matrix has the property

Example 6

The following are all identity matrices

\[
\begin{bmatrix}
1 & 0 & =1 & 2 \\
12 & 9 & 1 & -3 \\
=3 & 7 & 2 & 1 \\
2 & 0 & 3 & 4 \\
\end{bmatrix}
\]

8.4.3 The Zero matrix

The ‘Zero’ matrix is a matrix where every element in it is a zero.

Example 7: \[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\] is a zero matrix of order 2 \times 2.

\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\] is also a zero matrix of order 1 \times 2

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\] is a zero matrix of order 3 \times 3

8.4.4 Vectors

A vector is either a row or column matrix. A row matrix is a matrix that is made up of only one row. Its order is 1 \times n where n is the number of columns of a particular matrix. On the other hand, a column matrix is a matrix that is made up of only one column. It has order m \times 1 where m is the number of rows of a particular matrix.

Example 8

The following are vectors:

a) \[
\begin{bmatrix}
4 & 2 & 1 \\
\end{bmatrix}
\] is a row vector with order 1 \times 3
8.4.5 Scalar

In matrix notation, the term scalar refers to a constant (any real number)

Example 9

The following are normal matrices

\[ A = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 0 \\ 5 & 1 & 9 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 8 \\ 2 \\ 7 \\ 9 \\ 5 \end{pmatrix} \]

The following are scalars

2, 13, 6.5, -5, 0.37

Note: Scalars are constants.

8.5 USING MATRICES TO REPRESENT REAL LIFE DATA

Matrices can be used to represent real life data when data is capable of being viewed in two dimensions

Example 10

A company has two branches one located in Blantyre and the other in Mzuzu. There are 10 male and 6 female employees in Blantyre. There are 4 male and 2 female employees in Mzuzu. Represent the data in matrix form.

\[
\begin{pmatrix}
\text{Male} & \text{Female} \\
\text{Blantyre} & \begin{pmatrix} 10 & 6 \end{pmatrix} \\
\text{Mzuzu} & \begin{pmatrix} 4 & 2 \end{pmatrix}
\end{pmatrix}
\]

This is a 2 x 2 matrix.

Example 11

The table below shows the average number of passengers on three AXE Bus Service between two towns, Blantyre and Lilongwe. Also shown is the price of a ticket per person for each service:
### Required:

a) Find the total number of passengers departing at 11:30 am

b) Construct two Matrices to represent the information from the table.

#### Solution

a) Passengers departing at 11:30: \[12 + 40 + 70 = 122\]

b) Let the matrices be named \(N\) and \(P\) where \(N\) represents number of passengers and \(P\) represents prices.

\[N = \begin{pmatrix} 15 & 35 & 60 \\ 12 & 40 & 70 \\ 8 & 50 & 34 \end{pmatrix}\]

\[P = \begin{pmatrix} 4000 \\ 2500 \\ 2000 \end{pmatrix}\]

Note matrix \(P\) is a column matrix. It could have as well been written as a row matrix depending on the desired multiplication result.

### 8.6 Equality of Matrices

Matrices \(A\) and \(B\) are said to be equal if they are of the same order and corresponding elements are equal.

This is denoted \(A = B\).

Thus if \(A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\) and \(B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\), then \(A = B\) if \(a_{11} = b_{11}, a_{12} = b_{12}, a_{21} = b_{21}\) and \(a_{22} = b_{22}\)

#### Example 12

The following matrices are equal: \(A = \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}\) and \(B = \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}\)

i.e. \(A = B\).
Example 13

Given matrix M and N as outlined below:

\[
M = \begin{pmatrix} x & 3 \\ 1 & -4 \\ 3z & 7 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 2 & 3 \\ 1 & y \\ 12 & 7 \end{pmatrix}
\]

Find the values of \( x \), \( y \) and \( z \) for which \( M = N \).

**Solution:** Using equality of matrices, corresponding elements must be equal.

i.e. \( x = 2 \), \( y = -4 \) and \( 3z = 12 \) i.e. \( z = 4 \)

### 8.7 ARITHMETIC OPERATIONS ON MATRICES

#### 8.7.1 Addition and subtraction of matrices

When two matrices are being added or subtracted, it is the corresponding elements that get added or subtracted to form a new matrix. This means matrix addition or subtraction can only be defined if the matrices being added or subtracted are of the same order.

Thus if \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) and \( B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \)

Then \( A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \)

Note that it is the corresponding elements that add

**Example 14**

Given that

\[
A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 6 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}
\]

Find where possible

(a) \( A + B \)  
(b) \( C + D \)  
(c) \( A + D \)  
(d) \( B - A \)  
(e) \( B + A \)  
(f) \( D - B \)  
(g) \( A - B \)
Solution

(a) \[ A + B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 \!-\! (-2) & 3 \!+\! 1 \\ 2 \!+\! 3 & 4 \!+\! 2 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 5 & 6 \end{pmatrix} \]

(b) \[ C + D = \begin{pmatrix} 1 & 2 \\ 6 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \!+\! 5 & 2 \!+\! 6 \\ 6 \!+\! 2 & 0 \!+\! 1 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 8 & 1 \end{pmatrix} \]

(c) \[ A + B \] is not possible as \[ A \& D \] have different orders (\( A \) has order \( 2 \times 2 \) while \( D \) has \( 3 \times 2 \))

(d) \[ B - A = \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 \!-\! 1 & 1 \!-\! 3 \\ 3 \!-\! 2 & 2 \!-\! 4 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 1 & -2 \end{pmatrix} \]

(e) \[ B + A = \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 \!+\! 1 & 1 \!+\! 3 \\ 3 \!+\! 2 & 2 \!+\! 4 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 5 & 6 \end{pmatrix} \]

(f) \[ D - B \] is not possible. The two have different orders.

(g) \[ A - B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 \!-\! (-2) & 3 \!-\! 1 \\ 2 \!-\! 3 & 4 \!-\! 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix} \]

Notice that \( A + B = B + A \) and that \( A - B \neq B - A \).

We say that **Matrix Addition** is **Commutative** i.e. If \( A \) and \( B \) are matrices of the same order, \( A + B = B + A \). Notice also that Matrix subtraction is **NOT** commutative.

**Example 15**

Given matrix \( H, G \) and \( Q \) as outlined below

\[
M = \begin{pmatrix} x & 3 \\ 1 & -2 \\ 2 & 7 \end{pmatrix}, \quad N = \begin{pmatrix} 3 & 3 \\ 1 & y \\ 6 & 2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} y & 6 \\ 2 & 2x \\ 8 & 9 \end{pmatrix}
\]

Find the values of \( x \) and \( y \) if \( Q = M + N \).

**Solution**

Now \( Q = M + N \) implies
\[
\begin{pmatrix} y & 6 \\ 2 & 2x \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} x & 3 \\ 1 & -2 \\ 2 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 3 \\ 1 & y \\ 6 & 2 \end{pmatrix}
\]

Using equality of matrices, then

\[
3 + x = y \\
y - 2 = 2x
\]

By substitution, \(3 + x - 2 = 2x\)
Therefore \(3 + x - 2 = 2x\)
\(x = 1\)
So \(y = 3 + 1 = 4\)

8.7.2 Multiplication of matrices

Given two matrices \(A\) and \(B\). The product \(C = AB\) is defined only when the number of columns in \(A\) (the multiplicand) and the number of rows in \(B\) the multiplier are the same:

**Illustration**

If \(A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\) and \(B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}\)

Then the product matrix \(C = AB\) is defined because in \(A\), there are 3 columns and in \(B\) there are 3 rows. The product is found as follows:

\[
C = AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}
= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}
\]

**Note**

Row 1 elements in \(A\) multiply column 1 elements in \(B\). These are added to form element in row one column 1 in \(C\)

The same row 1 element then multiply corresponding elements in column 2 of \(B\). These are added to form the element in row 1 and column 2 of \(C\)
Row 2 elements in A multiply column 1 elements in B. These are added to form element in row 2 column 1 element in C.

The same row 2 elements then multiply corresponding elements in column 2 of B. These are added to form the element in row 2 and column 2 of C.

**Example 16**

Given that

\[
A = \begin{pmatrix} 4 & 9 \\ 3 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}.
\]

a) Find matrix C such that \( C = AB \).

b) Is \( AB = BA \)?

**Solution**

a) \( C = AB = \begin{pmatrix} 4 & 9 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 4 \times 3 + 9 \times 7 & 4 \times (-2) + 9 \times 4 \\ 3 \times 3 + (-6) \times 7 & 3 \times (-2) + (-6) \times 4 \end{pmatrix} = \begin{pmatrix} 75 & 28 \\ -33 & -30 \end{pmatrix} \)

b) \( BA = \begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} 4 & 9 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 3 \times 4 + (-2) \times 3 & 3 \times 9 + (-2) \times (-6) \\ 7 \times 4 + 4 \times 3 & 7 \times 9 + 4 \times (-6) \end{pmatrix} = \begin{pmatrix} 6 & 39 \\ 40 & 39 \end{pmatrix} \)

which is different from \( AB \) as calculated above.

**NOTE:** In normal algebra \( xy = yx \) but in matrix algebra \( AB \neq BA \)

**Example 17**

Given the two matrices \( M \) and \( N \) below. Which of the products, \( MN \) or \( NM \), is defined?

\[
M = \begin{pmatrix} 4 & 9 \\ 3 & -6 \end{pmatrix}, \quad N = \begin{pmatrix} 3 & -2 \\ 7 & 4 \\ 5 & 1 \end{pmatrix}
\]

Carry out the defined multiplication.
Solution

Following the rules of matrix multiplication, \( NM \) is defined because number of columns in \( N = \) number of rows in \( M \)

\[
NM = \begin{pmatrix}
3 & -2 \\
4 & -6 \\
6 & 39
\end{pmatrix}
\begin{pmatrix}
7 & 4 \\
5 & 1 \\
3 & -6
\end{pmatrix}
= \begin{pmatrix}
6 & 39 \\
40 & 39 \\
23 & 39
\end{pmatrix}.
\]

On the other hand, \( MN \) is not possible since the number of columns in \( M \) (2) is different from the number of rows (3) in \( N \).

Example 18

A business man purchases 20 tables, 80 chairs, and 2 water dispensers for his conference room. The respective unit prices are K35,000, K18,000 and K65,000.

a) Express the quantities and prices in terms of matrices \( Q \) and \( P \) in such a way that they can be multiplied.

b) Multiply the two matrices so that the interim result gives the expenditure in each item and the final result provides the total expenditure.

Solution

From the data the two matrices will be vectors.

a) For multiplication to be possible, one must be a column matrix and the other a row matrix:

\[
\begin{pmatrix}
280 \\
80 \\
2
\end{pmatrix}, \quad \begin{pmatrix}
35000 \\
18000 \\
56000
\end{pmatrix}
\]

b) Total expenditure:

\[
\begin{pmatrix}
35000 \\
18000 \\
56000
\end{pmatrix}
\times \begin{pmatrix}
20 & 80 & 2
\end{pmatrix}
= \begin{pmatrix}
20 \times 35000 + 80 \times 18000 + 2 \times 56000
\end{pmatrix}
= (2,252,000)
\]

i.e. total expenditure is K2,252,00

Example 19

Product X has fixed costs of K12,000 and variable costs of K1,200 per product, product Y has fixed cost of K7,000 and variable cost of K1,600 per product.
Required:

a) If \( C = \begin{pmatrix} 12000 & 1200 \\ 7000 & 1600 \end{pmatrix} \) and \( Q = \begin{pmatrix} 1 \\ q \end{pmatrix} \), where \( C \) is the cost coefficient matrix, evaluate the matrix product \( CQ \) and interpret.

b) Given that product X and Y sell at K1,800 and K2,000 respectively:

i. Write down the revenue matrix \( R \)

ii. Evaluate the matrix \( RQ - CQ \)

c) Putting \( P = RQ - CQ \) and \( A = (1 \ 0) \), solve the matrix equation \( AP = BP \) and interpret the value of \( q \) obtained.

Solution

a) Note that this is matrix multiplication: \( C \) has 2 columns and \( Q \) has 2 rows. So multiplication is well defined.

\[
CQ = \begin{pmatrix} 12000 & 1200 \\ 7000 & 1600 \end{pmatrix} \begin{pmatrix} 1 \\ q \end{pmatrix} = \begin{pmatrix} 12000 + 1200q \\ 7000 + 1600q \end{pmatrix}
\]

Results:

The first element is a cost function of product X and the second element is a cost function of product Y.

b) Revenue coefficient matrix \( R = \begin{pmatrix} 0 & 1800 \\ 0 & 2000 \end{pmatrix} \)

This is prompted by the fact that when multiplied by \( Q \) there will be no notion of “fixed revenues”.

\[
RQ = \begin{pmatrix} 0 & 1800 \\ 0 & 2000 \end{pmatrix} \begin{pmatrix} 1 \\ q \end{pmatrix} = \begin{pmatrix} 0 + 2000q \\ 0 + 1800q \end{pmatrix} = \begin{pmatrix} 2000q \\ 1800q \end{pmatrix}
\]

\[
P = RQ - CQ = \begin{pmatrix} 2000q \\ 1800q \end{pmatrix} - \begin{pmatrix} 12000 + 1200q \\ 7000 + 1600q \end{pmatrix} = \begin{pmatrix} 600q - 12000 \\ 1200q - 7000 \end{pmatrix}
\]

The element in the matrix \( P \) are profit functions for product X and Y respectively.

Given that \( A = (1 \ 0) \) and \( B = (0 \ 1) \).
Then \( AP = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 600q - 12000 \\ 200q - 7000 \end{pmatrix} = (600q - 12000) \)

and \( BP = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 600q - 12000 \\ 200q - 7000 \end{pmatrix} = (200q - 7000) \)

Now \( AP = BP \Rightarrow 600q - 12000 = 200q - 7000 \)
\( \Rightarrow 600q - 200q = 12000 - 7000 \)
\( i.e. \ 400q = 5000 \)
\( q = 12.5 \)
\( q = 12.5 \) is the point at which profits on \( x \) and profits on \( y \) are the same.

### 8.7.3 Scalar multiplication
Scalar Multiplication is the multiplication of a matrix by a scalar (a constant). This is done by multiplying each element of the matrix by a scalar.

**Example 20**

Let \( A = \begin{pmatrix} 4 & 9 \\ 3 & -6 \end{pmatrix} \), \( B = \begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix} \) and \( C = \begin{pmatrix} -1 & 10 \\ 0 & 3 \end{pmatrix} \)

Evaluate the following:

a) \( 3A \)
b) \( 2B - 4C \)
c) \( 2C - A + 3B \)

**Solution**

a) \( 3A = 3 \begin{pmatrix} 4 & 9 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 3 \times 4 & 3 \times 9 \\ 3 \times 3 & 3 \times (-6) \end{pmatrix} = \begin{pmatrix} 12 & 27 \\ 9 & -18 \end{pmatrix} \)

b) \( 2B - 4C = 2 \begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix} - 4 \begin{pmatrix} -1 & 10 \\ 0 & 3 \end{pmatrix} \)
\( = \begin{pmatrix} 2 \times 3 & 2 \times (-2) \\ 2 \times 7 & 2 \times 4 \end{pmatrix} - \begin{pmatrix} 4 \times (-1) & 4 \times 10 \\ 4 \times 0 & 4 \times 3 \end{pmatrix} \)
\( = \begin{pmatrix} 6 & -4 \\ 14 & 8 \end{pmatrix} - \begin{pmatrix} -4 & 40 \\ 0 & 12 \end{pmatrix} \)
\( = \begin{pmatrix} 10 & -44 \\ 14 & 28 \end{pmatrix} \)
\[
= \begin{pmatrix}
6 - (-4) & -4 - 40 \\
14 - 0 & 8 - 12
\end{pmatrix}
\]
\[
= \begin{pmatrix}
6 + 4 & -4 - 40 \\
14 - 0 & 8 - 12
\end{pmatrix}
\]
\[
= \begin{pmatrix}
10 & -44 \\
14 & -4
\end{pmatrix}
\]

\[
c) \quad 2C - A + 3B = 2 \begin{pmatrix}
-1 & 10 \\
0 & 3
\end{pmatrix} - \begin{pmatrix}
4 & 9 \\
3 & -6
\end{pmatrix} + 3 \begin{pmatrix}
3 & -2 \\
7 & 4
\end{pmatrix}
\]
\[
= \begin{pmatrix}
2 \times (-1) & 2 \times 10 \\
2 \times 0 & 2 \times 3
\end{pmatrix} - \begin{pmatrix}
4 & 9 \\
3 & -6
\end{pmatrix} + 3 \begin{pmatrix}
3 \times 3 & 3 \times (-2) \\
3 \times 7 & 3 \times 4
\end{pmatrix}
\]
\[
= \begin{pmatrix}
-2 & 20 \\
0 & 6
\end{pmatrix} - \begin{pmatrix}
4 & 9 \\
3 & -6
\end{pmatrix} + \begin{pmatrix}
9 & -6 \\
21 & 12
\end{pmatrix} = \begin{pmatrix}
-2 - 4 + 9 & 20 - 9 - 6 \\
0 - 3 + 21 & 6 + 6 + 12
\end{pmatrix}
\]
\[
= \begin{pmatrix}
3 & 5 \\
18 & 24
\end{pmatrix}
\]

### 8.7.4 Determinant of a matrix

**a) Determinant of a 2 x 2 matrix**

Determinant of matrix, denoted by two vertical lines binding the matrix

Given a matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

The determinant of \( A \), denoted by \( |A| \) or \( \text{det}(A) \) is \( \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \).
Example 22

Given that $D = \begin{pmatrix} 3 & 2 \\ 1 & -2 \end{pmatrix}$, find $|D|$.

Solution

Determinant of $D$, $|D| = \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} = 3 \times (-2) - 2 \times 1 = -6 - 2 = -8$.

b) Determinant of a $3 \times 3$ matrix or higher

The easiest way of finding the determinant of a $3 \times 3$ matrix (or higher order) is to use sarrus' rule:

Step 1: Given a matrix, create an augmented matrix by writing the same matrix side by side.

Step 2: Multiply the elements of each lead diagonal (without repetition). There will be a product for each set of elements in the diagonal. Add these products.

Step 3: Repeat step 2 for the non-lead diagonals.

Step 4: Subtract the sum in step 3 from the sum in step 2, this is the determinant.

Example 23

Find the determinant of $K = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

Solution

Step 1: Create an augmented matrix by writing the same matrix side by side.

Step 2: Multiply the elements of each lead diagonal

The lines show elements in three lead diagonals.

The products are: $3 \times 1 \times (-2) = -6$, $2 \times (-1) \times 3 = -6$ and $1 \times 1 \times 1 = 1$
The sum is \((-6) + (-6) + 1 = -11\)

Step 3: Repeat step 2 for the non-lead diagonals

\[
\begin{array}{ccc}
3 & 2 & 1 \\
1 & 1 & -1 \\
3 & 1 & -2 \\
\end{array}
\quad
\begin{array}{ccc}
3 & 2 & 1 \\
1 & 1 & -1 \\
3 & 1 & -2 \\
\end{array}
\]

The lines are on the non-leading diagonals.
The products are \(3 \times (-1) \times 1 = -3; \ 2 \times 1 \times (-2) = -4; \ 1 \times 1 \times 3 = 3\)
The sum is \(-3 + (-4) + 3 = -4\)

Step 4: Find the determinant of \(K\) This \((-11 - (-4)) = -7\)

Note: This process can be used for a 4 x 4 matrix or higher.

8.7.5 The inverse of a matrix

A. Definition
The inverse of a square matrix \(A\) denoted by \(A^{-1}\) is a matrix defined as
\(A^{-1} = AA^{-1} = A^{-1}A = I; \ I\) being an identity matrix.

B. Finding the inverse of a 2 x 2 matrix

Given a matrix \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\)
The inverse of \(A = A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\)

\[
= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

Note: As we cannot divide by 0, we insist that \(|A| = ad - bc \neq 0\). It turns out that \(|A| = 0\), we shall conclude that the inverse does not exist.

Example 24

Let \(A = \begin{pmatrix} 7 & -3 \\ 9 & -3 \end{pmatrix}\), find the inverse of \(A\).
Solution

\[
A^{-1} = \frac{1}{|A|} \begin{pmatrix} -3 & 3 \\ -9 & 7 \end{pmatrix} = \frac{1}{7 \times (-3) - (-3) \times 9} \begin{pmatrix} -3 & 3 \\ -9 & 7 \end{pmatrix} \\
= \frac{1}{6} \begin{pmatrix} -3 & 3 \\ -9 & 7 \end{pmatrix}
\]

We can check if this is indeed the inverse matrix by finding the product \( AA^{-1} \)

\[
AA^{-1} = \begin{pmatrix} 7 & -3 \\ 9 & -3 \end{pmatrix} \frac{1}{6} \begin{pmatrix} -3 & 3 \\ -9 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I
\]

Example 25

Let \( D = \begin{pmatrix} 3 & -2 \\ -6 & -4 \end{pmatrix} \). Find \( D^{-1} \).

Solution

Now we need \( |D| = 3 \times (-4) - (-2) \times (-6) = -12 + 12 = 0 \). Since \( |D| = 0 \), we conclude that \( D^{-1} \) does not exist i.e. matrix \( D \) does not have an inverse.

8.8 SOLVING SYSTEMS OF SIMULTANEOUS LINEAR EQUATIONS USING MATRICES

8.8.1 Overview.

Matrices are quite handy tools for solving systems of linear equations. Under matrix algebra it is possible to solve a system of linear equations. This can be done by:

a) The inverse method.

b) Using the Cramer’s rule

The latter is shorter especially for 3 x 3 matrices than those of higher order.

8.8.2 Solution by inverse method

Consider the system of linear equations

\[
\begin{align*}
& a_{11}x + a_{12}y = r \\
& a_{21}x + a_{22}y = s.
\end{align*}
\]

This system of linear equations can be expressed in matrix form as follows:

\[
AX = B,
\]
where \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}\) and \( B = \begin{pmatrix} r \\ s \end{pmatrix}\).

Thus \( AX = B \Rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix} \)

Note that if matrix multiplication is carried out and equality of matrices used, the left side and the right yield the system of linear equations:

\[
\begin{align*}
a_{11}x + a_{12}y &= r \\
a_{21}x + a_{22}y &= s.
\end{align*}
\]

If the inverse of \( A \) is defined, the system of the equations can be solved as follows:

Given that \( AX = B \) and that the inverse, \( A^{-1} \), exists.

Then \( A^{-1}AX = A^{-1}B \)
\[
I X = A^{-1}B
\]
\[
\text{or} \quad X = A^{-1}B,
\]

Recall that \( X = \begin{pmatrix} x \\ y \end{pmatrix} \) i.e. \( x \) and \( y \). The solution therefore is actually found by multiplying the inverse of \( A \) and matrix \( B \).

**Example 26**

Solve the simultaneous equations \( y = 3x + 5; \quad \text{and} \quad 2y + 3x = 28 \) by

a) Ordinary algebraic method
b) By matrix method (using the inverse)

**Solution**

a) By ordinary algebra

Substitute \( y \) into equation 2:

\[
\begin{align*}
2(3x + 5) + 3x &= 28 \\
6x + 10 + 3x &= 28 \\
9x &= 18 \\
x &= 2
\end{align*}
\]

Substitute \( x \) into equation 1:

\[
y = 3(2) + 5 = 11
\]
b) Using matrix algebra (the inverse):

The equations: \( y = 3x + 5 \); and \( 2y + 3x = 28 \) can be rearranged as:
\[
-3x + y = 5 \\
3x + 2y = 28
\]
and be written as
\[
\begin{pmatrix}
-3 & 1 \\
3 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
5 \\
28
\end{pmatrix}
\]
i.e. \( AX = B \)

If \( A^{-1} \) exists, then \( X = A^{-1}B \).

Now \( A^{-1} = \frac{1}{|A|} \begin{pmatrix} 2 & -1 \\ -3 & -3 \end{pmatrix} = \frac{1}{-6-(-9)} \begin{pmatrix} 2 & -1 \\ -3 & -3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -3 & -3 \end{pmatrix} \)

Then \( X = A^{-1}B = \frac{1}{-9} \begin{pmatrix} 2 & -1 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} 5 \\
28
\end{pmatrix} = \frac{1}{-9} \begin{pmatrix} -18 \\
-99
\end{pmatrix} = \begin{pmatrix} (-1/9) \times (-18) \\
(-1/9) \times (-99)
\end{pmatrix} = \begin{pmatrix} 2 \\
11
\end{pmatrix} \)

So \( \begin{pmatrix} x \\
y
\end{pmatrix} = \begin{pmatrix} 2 \\
11
\end{pmatrix} \)

Using equality of matrices, \( x = 2, \ y = 11 \)

8.8.3 Solution using Cramer’s Rule

Given a system of linear equations, Cramer's Rule is a handy way to solve for just one of the variables without having to solve the whole system of equations.

Cramer’s Rule uses determinants as illustrated below.

Consider the system of linear equations:
\[
a_{11}x + a_{12}y = r \\
a_{21}x + a_{22}y = s.
\]

Provided \( |A| \neq 0 \), then \( x = \frac{\begin{vmatrix} r & a_{12} \\ s & a_{22} \end{vmatrix}}{|A|} \), \( y = \frac{\begin{vmatrix} a_{11} & r \\ a_{12} & s \end{vmatrix}}{|A|} \)
Example 27

Use the cramer’s rule to solve the system of equations

\[ y = 3x + 5; \text{ and } 2y + 3x = 28 \]

Solution

The equations: \[ y = 3x + 5; \text{ and } 2y + 3x = 28 \] can be rearranged as:

\[
\begin{align*}
-3x + y &= 5 \\
3x + 2y &= 28
\end{align*}
\]

and be written as

\[
\begin{pmatrix}
-3 & 1 \\
3 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
5 \\
28
\end{pmatrix}
\text{ i.e. } AX = B
\]

Now

\[
\begin{vmatrix}
-3 & 1 \\
3 & 2
\end{vmatrix}
= (-3) \times 2 - 1 \times 3 = -9
\]

Then

\[
\begin{pmatrix}
5 & 1 \\
28 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
3 & 5 \\
-9 & 28
\end{pmatrix}
\]

In summary, the solutions (variable by variable) are found as follows:

Step 1: Find the determinant of the coefficient matrix A

Determinant of Matrix A as calculated above is -9

Step 2: Create a matrix \(A_x\) by replacing the \(x\) column in matrix A by the solution vector (i.e. elements of the right hand side). Find the determinant of this new matrix.

Step 3 Create another matrix \(A_y\) by replacing the \(y\) column in A with the solution vector

Using Cramer’s Rule, solve the following system of linear equations:

\[
\begin{align*}
2x + \ y + z &= 3 \\
x - y - z &= 0 \\
x + 2y + z &= 0
\end{align*}
\]
Solution

Now \( A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{pmatrix}, \) Need \( A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \)

Then

\[
|A| = 2 \times (-1) \times 1 + 1 \times (-1) \times 1 + 1 \times 2 - (1 \times (-1) \times 1 + 2 \times (-1) \times 2 + 1 \times 1 \times 1)
\]

\[
= -2 - 1 + 2 + 1 + 4 - 1 = 3.
\]

The solution vector is \( B = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \).

Then

\[
x = \frac{|A_x|}{|A|} = \frac{3 \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}}{1} = 1
\]

\[
y = \frac{|A_y|}{|A|} = \frac{2 \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}}{1} = -6
\]

\[
z = \frac{|A_z|}{|A|} = \frac{2 \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}}{1} = 3
\]

So \( x = 1, \ y = -6, \ z = 3. \)
CHAPTER SUMMARY

A matrix is a table of values conveying numerical information. Having defined what matrices are, the chapter looked at different types of matrices and presented algebraic operations associated with matrices. Under types of matrices, examples presented included the zero matrix, identity matrix, square matrices, row matrices and column matrices. On the other hand, algebraic operations included addition, subtraction, multiplication, determinants and inverses (in matrix algebra, there is no division!).

In matrix addition or subtraction, matrices have to be of identical size and involves adding or subtracting corresponding elements. Under multiplication, the chapter presented scalar multiplication, which involved multiplying every element of the matrix by a scalar, and matrix multiplication which involved two matrices in which the number of columns of one matrix is equal to the number of rows of the other, otherwise it will not be possible to perform matrix multiplication. In addition to this, the chapter also presented the concept of determinant which is cardinal when finding inverses of matrices and solving systems of linear equations.

Matrices are incredibly useful algebraic structures that have numerous applications in mathematics and other sciences. Some of them merely take advantage of the compact representation of a set of numbers in a matrix. In this chapter, matrices were used in solving systems of linear equations. The methods for solving systems of linear equations were the matrix inversion method and Cramer’s Rule. The method of matrix inversion involves multiplying the right hand side by the inverse of the coefficient matrix while Cramer’s Rule used determinants.

END OF CHAPTER EXERCISES

1. Consider the following matrices: 

\[
A = \begin{pmatrix}
12 & -4 \\
8 & 3 \\
1 & 6
\end{pmatrix}, \quad \quad B = \begin{pmatrix}
5 & -2 \\
3 & 3 \\
-1 & 7
\end{pmatrix} \quad \text{and} \quad \quad C = \begin{pmatrix}
2 & 0 & 3 \\
1 & -3 & 1
\end{pmatrix}
\]

Find the following, where possible:

a) \( A - B \)

b) \( A + B \)

c) \( 3A \)

d) \( 3A - 2B \)

e) \( AC \)

f) \( CA \)

g) \( BC \)

2. Given two matrices 

\[
A = \begin{pmatrix}
2 & 5 \\
3 & 2 \\
1 & 0
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
3 & -2 \\
4 & 1
\end{pmatrix}
\]

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a) Which of these, AB or BA is defined?
b) Carry out the defined multiplication.

3. Let \[
A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}
\]

Find the matrices
(i) \(A + B\)
(ii) \(B + C\)
(iii) \((A + B) + C\)
(iv) \(A + (B + C)\)

What conclusion can you draw from (iii) and (iv)?

4. (a) Evaluate \(AB\), given that matrices \(A\) and \(B\) are defined as:
\[
A = \begin{pmatrix} 2 & -1 & 3 \\ 5 & 7 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -4 \\ 1 & 4 \\ 5 & -1 \end{pmatrix}
\]

(b) The table below shows the number of passengers on three AXE Bus Services between two towns Blantyre and Lilongwe and the price of a ticket per person for each service:

<table>
<thead>
<tr>
<th>Departure time</th>
<th>Comfort</th>
<th>Excursion</th>
<th>Economy</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.30</td>
<td>15</td>
<td>35</td>
<td>60</td>
</tr>
<tr>
<td>11.30</td>
<td>12</td>
<td>40</td>
<td>70</td>
</tr>
<tr>
<td>16.00</td>
<td>8</td>
<td>50</td>
<td>34</td>
</tr>
<tr>
<td>Price (K)</td>
<td>4,000</td>
<td>2,500</td>
<td>2,000</td>
</tr>
</tbody>
</table>

Required:
(i) Find the total number of passengers departing at 11:30 a.m.
(ii) Construct two matrices to represent the information from the table.
(iii) Use the matrices from part (ii) to obtain the total revenue from each Departure time.

5. Find \(x\) and \(y\) in each of the following
(a) \(2x + 3y = 8\)
(b) \[
\begin{pmatrix} x + 1 \\ y - 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix}
\]
A housewife makes the following purchases during one week:

Monday: 2 packets of milk and 1 loaf of bread;
Tuesday: 3 litres of milk;
Wednesday: 1 litre of milk and 2 loaves of bread;
Thursday: 2 loaves of bread;
Friday: 3 litres of milk;
Saturday: 2 litres milk and 2 loaves of bread;
Sunday: 3 loaves of bread.

Arrange her purchases in a matrix with two rows and seven columns.

Solve each of the following equations for $A$, where $A$ is a $2 \times 2$ matrix.

(a) \[ 2A = \begin{pmatrix} 4 & 10 \\ 6 & 14 \end{pmatrix} \]

(b) \[ \begin{pmatrix} -2 & 6 \\ 5 & 2 \end{pmatrix} + 3A = \begin{pmatrix} 19 & 18 \\ 20 & 11 \end{pmatrix} \]

Solve the following systems of equations using the inverse method:

\[
\begin{align*}
\begin{cases}
y = 2x + 1 \\
y = -2x + 9
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
3x + 4y = 5 \\
x - y = 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
3x + 4y = -2 \\
5x + 3y = 4
\end{cases}
\end{align*}
\]

Solve the following using the Cramer’s rule:

(a) \[
\begin{align*}
\begin{cases}
3x + 4y = 5 \\
2x - y = 0
\end{cases}
\end{align*}
\]

(b) \[
\begin{align*}
\begin{cases}
3x - y + z = 5 \\
2x + 2y + 3z = 4 \\
x + 3y - z = 11
\end{cases}
\end{align*}
\]

(c) \[
\begin{align*}
\begin{cases}
2x + 5y + z = 8 \\
3x + y - 2z = 7 \\
2x + 10y + 2z = 16
\end{cases}
\end{align*}
\]

(d) \[
\begin{align*}
\begin{cases}
3x + 3y + 4z = 1 \\
5x + 9y + 17z = 4 \\
3x + 5y + 9z = 2
\end{cases}
\end{align*}
\]
10. Product x has fixed costs of K12,000 and variable costs of K1,200 per product; product y has fixed costs of K7,000 and variable costs of K1,600 per product.

a) If \( C = \begin{pmatrix} 12000 & 1200 \\ 7000 & 1000 \end{pmatrix} \) and \( Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), where \( C \) is the cost coefficient matrix, evaluate the matrix product \( CQ \), and interpret.

b) Given that products x and y sell at K1,800 and K2,000 respectively:
   (i) Write down the revenue coefficient matrix, \( R \) and
   (ii) Evaluate the matrix \( RQ - CQ \) and explain its significance.

b) Putting \( P = RQ - CQ \), \( A = \begin{pmatrix} 1 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 1 \end{pmatrix} \), solve the matrix equation \( AP = BP \) and interpret the value of \( q \) obtained.
CHAPTER 9 DATA AND DATA COLLECTION

LEARNING OBJECTIVES

At the end of this chapter the student should be able to:

i. Distinguish between data and information
ii. Classify data
iii. Distinguish between data collection methods and select a suitable method
iv. Distinguish between different types of data
v. State stages in statistical investigation

9.0 INTRODUCTION

This chapter is concerned with the various methods employed in choosing the subjects for an investigation and the different ways that exist for collecting data. The advantages and disadvantages of the data collection methods have been outlined.

9.1 DATA AND INFORMATION

9.1.1 Definitions

Data is a term used to mean raw facts which one can use to produce information which can address a question or explain a situation.

Information is processed data that is meaningful to the user.

Processing of data can be any of the following.

a. Classification
b. Computation
c. Sorting
d. Categorisation

9.1.2 Classification of data

Overview

Data can be classified into many groups. The bases include nature of measurement and source.

If nature of measurement is considered, data can be classified as discrete, continuous, qualitative or quantitative, while if the source of data is taken into account, terms like primary or secondary data can be used.
It should be noted that some of these classifications are not mutually exclusive, they do overlap. For example, quantitative data can be secondary or primary.

**Quantitative and Qualitative**

Quantitative data are numerical measurements expressed in terms of numbers and the numbers stand for specific values. They are not mere labels.

**Example 1**
- Balance at a bank: K2344.2
- Number of students in a class: 35
- Height of a person: 1.8m

Qualitative data comprises labels, opinions or expressions of conclusions. As opposed to quantitative data which is in numeric form, qualitative data is expressed by means of a natural language descriptions. Sometimes qualitative data is referred to as “categorical” data.

**Example 2**
- Today’s weather: “bad”
- Level of attendance: “high”
- Comment on Project: “successful”
- Grade in Maths at MSCE: “9”
- Expressions of size: “small, medium and large”
- Sociologist conclusion: “poverty fuels spread of HIV”
- Storm category: “5”

Note that use of numbers in data does not necessarily make that data quantitative.

**Example 3:** The number at the back of a football player: 11

<table>
<thead>
<tr>
<th>Scores representing opinions:</th>
<th>1 (poor)</th>
<th>5 (Excellent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Storm category</td>
<td>5 (bad)</td>
<td></td>
</tr>
</tbody>
</table>

The numbers above are only labels (much as some can be described as some form of ranking). These numbers can be changed without changing the meaning or what it represents. For instance, a player wearing jersey 9 can put on jersey 12. He remains the same person. Excellent” can be a 4 and other time a 5 can be used. A 9 is a label for “fail” on the Malawi School Certificate level. In other exam situations a “0” means fail. Contrast this with a bank balance of MK17.00. If the entry is MK0.00, the meaning is very different.
**Discrete and continuous**

Discrete data is data relating to “countable variables.” The data is mainly in the form of integers.

Examples: 2 cars, 5 people, 272 tablets of soap

Continuous data is data which is part of a continuum. The user simply decides on a limit.

Items whose data is normally continuous include

Age, Time, Length, Height, Weight

Continuous data is characterized by the presence of fractions in its measurement, although some will be expressed as whole numbers.

Examples

i. The age of a child: 5.7 yrs
ii. The distance between towns: 23.8 km
iii. A bag of maize: 50 kg

**Primary and Secondary Data**

Primary data means original data that have been collected specially for the purpose in mind. Primary data can also be defined as data collected from an original source.

It means when an authorized organization, an investigator or an enumerator collects the data for a specific purpose and from an original source himself or with the help of an institution or an expert then the data thus collected are called primary data.

Usually primary data is collected through surveys, observations or participation. Various tools for data collection (questionnaires, schedules or even check lists) exist.

*Research where one gathers this kind of data is referred to as *field research*.*

Secondary data are data that have been collected for another purpose. Examples of Secondary data include

Data from magazine, newspapers, books, or any publication.

*Research where one gathers this kind of data is referred to as *desk research*.*

**9.1.3 Other important terms**

**Parameter**

A parameter is a constant or variable term in a function that determines the specific form of the function.
Statistic
A statistic can be any specific value of interest that can be collected in relation to data of interest. Thus a value of age can be a statistic and so is the population of a town. Alternatively, a statistic is a value obtained from a variable entering into a mathematical form of any distribution such that the possible values of the variable corresponding to different distributions can be determined. Thus an average calculated using some function or formula is a statistic.

Stages in Statistical investigation

A statistical investigation involves a number of stages:

i. Definition of the problem or issue
ii. Collection of relevant data
iii. Classification and analysis of the collected data
iv. Presentation of the results

Before the collection of data starts, there are important points to consider when planning a statistical investigation.

Preliminary Considerations

It is extremely important to consider the following points before you start collecting data:

- **Aim**
  This is necessary in order to ensure that nothing important is omitted from the enquiry and the effort is not wasted by collecting irrelevant data. Many people fail to think clearly about the problem being investigated. You must write down the objective of the survey, stating exactly what information you want to get out of the data that you are planning to collect, then you will collect only relevant data.

- **Units**
  It is essential that on what units to use before data collection starts. The results must appear in comparable units for any analysis is to be valid. The choice of units should be influenced by the possible need to compare sets of data collected from different sources. Frequently data will be collected by a number of people all using different units. Some conversion factors would therefore be needed.

- **Scope of the enquiry**
  No investigation should be undertaken without defining the field to be covered. Should the study cover all the departments of the organisation or only some. Should it cover more than one organisation.

- **Accuracy of the data**
  If the level of accuracy is not defined beforehand then it is difficult to know the amount of detail to be collected. To what degree of accuracy is data to be recorded?. For example, are ages of individuals to be given to the nearest year or month or as the number of completed years?. If some of the data is to come from measurement, then the accuracy of the measuring instrument will affect the accuracy of the results. The degree of precision required in an estimate might affect the amount of data that we need to collect. In
general, the more precisely we wish to estimate a value, the more readings we need to take. If the level of accuracy is stated beforehand it is easier to estimate the cost of the data collection.

9.1.4 Requirement of Statistical data

Attributes of good data

For data to be classified as of high quality, it must have the following attributes:

a) Accuracy

Data should be sufficiently accurate for the intended use. For example day to day operations planning require that data should be detailed to the last unit of measurement (e.g. a car costing K7,850,330.26). Whereas data for long term planning requires rounded figures (e.g. the same car can be quoted as costing 8 million kwacha). Data should be captured only once, although it may have multiple uses. Data should be captured at the point of activity.

b) Validity

 Validity refers to correctness and reasonableness of data. Correctness looks at connection to the subject matter data is to relate to and reasonableness includes issues like age falling within expected ranges, account or phone numbers having certain digits and within specific ranges, and names spelt correctly. Data should also be correct and be recorded in compliance with relevant requirements, including the correct rules or definitions.

c) Reliability

Data should reflect stability and consistency in collection processes (methods), across collection points and over time.

d) Timeliness

Data should be captured as quickly as possible after the event or activity and must be available for the intended use within a reasonable time period. Data must be available quickly and frequently enough to support information needs and to influence service or management decisions.

e) Relevance

Data captured should be related to the purposes for which it is collected. This will require a periodic review of requirements to reflect changing needs.

f) Completeness

Data should be able to address all needs for which it is collected. This means data requirements should be clearly specified based on the information needs of the organisation and data collection processes matched to these requirements.
9.2 DATA COLLECTION METHODS

9.2.0 Overview

The following are the most common ways of collecting primary data.

Data can be collected in a number of ways. The following are most common methods:

i. Personal interviews
ii. Observations
iii. Questionnaires.
iv. Use of published statistics

The choice of a data collection method will be influenced by a number of factors including cost, speed, volume of data, type of data, expertise, etc.

9.2.1 Personal Interviews

Definition

These are face to face interviews where the interview collects data by asking questions directly to the respondent.

Personal interviews can be Structured, Unstructured and Semi-structured

*Structured*

These are interviews guided by a structured questionnaire. One must have clear objectives of an enquiry and develop a questionnaire with questions which are logically arranged. The interviewer uses these questions and the flow to collect data.

The aim of structured interview is to ensure that each interview employs exactly the same questions and in the same order. This ensures that answers can be reliably aggregated and that comparisons can be made with confidence between sample subgroups or between different survey periods.

*Unstructured*

Unstructured Interviews are a method of interviews where the interview has a topic to explore and a set of key questions (not necessarily in an order. These questions can be changed or adapted to meet the respondent's intelligence, understanding or belief. Unlike a structured interview they do not offer a limited, pre-set range of answers for a respondent to choose, but instead they encourage listening to how each individual person responds to the question.

The interview mostly proceeds in a “friendly manner”. However, it may lack the reliability and precision of the structured interview.
Semi structured – it is a bit of structured and unstructured.

Semi-structured interviews are in-depth interviews. They take on features of both structured and semi-structured interviews in that the interviewer uses predetermined questions and follows a pattern while being flexible enough to probe deeper depending on responses and at the same time allowing the interviewee the options to take different paths and explore different thoughts, feelings. The interview has however got to control the interview and should bring the interview back to the subject matter.

Advantages and Disadvantages of Personal Interviews

Personal interviews have the following advantages:

a. The interviewer can achieve a high response rate.
b. The method enables flexibility in terms of interpreting questions to the respondent.
c. There is usually better depth of data.

The disadvantages are that:

a. Responses may be distorted by wrong emphasis on the part of the interviewer.
b. The interviewer may lose control
c. Personal interviews may be expensive because travel may be involved.

9.2.2 Observations

Definition

In this method data is collected through participation and taking measurement then documenting the results. Sometimes observations can be carried out through experiments.

Advantages and Disadvantages of Observations

The method has the following advantages.

- A high degree of depth of the data can be achieved
- Data is usually very accurate because the results are a first hand account of a subject matter.

The disadvantages are:

- Cost of observations can be high
- Observations require a lot of time.
- Some situations may not render themselves easy to experimentation.

9.2.3 Questionnaires

Definition

This is where one designs a questionnaire and either hands it in person or mails to the respondent. The respondent completes the questionnaire and sends it back to the one collecting data.
Principles of good questionnaire design

i. Keep the questionnaire as short as possible.
ii. Keep the individual questions as simple as possible.
iii. Questions should not be capable of different interpretations.
iv. Avoid questions that require the respondent to make calculations.
v. Arrange questions in a logical manner.
vi. Avoid leading questions.
vii. If possible use closed questions.
viii. Explain the purpose of the investigation so as to encourage people to respond.

Advantages and disadvantages of self completion questionnaires

Advantages include:

i. Low cost. If mailed the major costs include stamps and paper
ii. Convenience (the respondent can fill the question at any time)
iii. Depth of data because of the time allowed, detailed questions can be asked.

Major disadvantages of self completion Questionnaire are:

i. High non response rate. The very fact of trying to answer the questionnaires
ii. Distortion of data due to varying interpretation.
iii. Difficulty to come up with a good questionnaire.

9.2.4 Published Statistics

Sometimes we may be attempting to solve a problem that does not require us to collect new data but to reassemble and reanalyse data which has already been collected by someone else for some other purpose. We can make good use of the amount of statistical data published by government, nationalised industries, chambers of commerce and so on. When using this method it is particularly important to be clear on the definition of the terms and units used and the accuracy of the data. The information must be reliable and up-to-date.

The information you require may not be found in one source but parts may appear in several different sources.

Advantages of published statistics

i. The cost is low in acquiring the data since the data is already available.
ii. The information can easily be accessed.
iii. It can be used to clarify the research questions.
iv. It can be used to facilitate in aligning the focus of primary research at large scale.

Disadvantages

i. It is time consuming in searching for appropriate data
ii. The definitions used for variables and units may not be the same as you wish to use.
iii. The data may be not be presented in the form the researcher needs
iv. The data may not give full version of the data as required by the researcher
v. There might be a problem of time, as data collected five years ago may not match with data collected now.
vi. The data is largely self-governed therefore the data must be scrutinised closely

CHAPTER SUMMARY

In this chapter you have leaned:

- Definition of data and information.
- Classification of data into quantitative and qualitative, for quantitative discrete and continuous.
- The difference between primary and secondary data.
- Attributes of good information.
- Data collection methods including their advantages and disadvantages.

END OF CHAPTER EXERCISES

1. Explain the meaning of the following in the context of statistics and in each case give an example:
   a) Quantitative variable.
   b) Qualitative variable.
   c) Statistic.
   d) Parameter

2. Classify the following as qualitative or quantitative data:
   a) Employees salaries on the payroll
   b) Employees, employment numbers on the same payroll
   c) Electricity units purchased for a prepaid meter
   d) Competition winners telephone numbers copied from a newspaper.
   e) Number of jobs handled by a garage
   f) The number at the back of a marathon runner
   g) Receipt numbers written out by an auditor from a sales file.

1. Classify the data in question 1 as primary or secondary.

2. An officer in the Ministry of Agriculture is asked to evaluate the benefits of the fertiliser subsidy programme to the ordinary farmer. What data Collection method will he/she use and why.

3. Using principles of a good questionnaire, design a questionnaire to be used by a human resources manager to collect data on employees responsibilities at work in the light of ones qualification and experience.
CHAPTER 10  SAMPLING

LEARNING OBJECTIVES

At the end of this chapter, the student should be able to

i. Distinguish between sampling and taking a census
ii. Outline the major sampling techniques
iii. Be able to suggest a suitable sampling technique for a situation and take a sample using any of the major techniques.

10.0 INTRODUCTION

In data collection, an ideal situation is to canvas or collect data from all elements of a population. This is taking a census (or complete “count”) of a population. However this is not always practical for many reasons which include.

- Cost: censuses are vast undertakings requiring large amounts of equipment, materials, and human resource
- Time: Censuses require time in planning, data Collection and analysis.
- Depth of data: Because of the size of the undertaking, it is not possible to ask in depth questions

Instead of a census, a sampling is used. A sample is a group of selected elements of the population. Sampling refers to the process of selecting the elements from the population.

For many enquiries, sampling is preferred because of costs are lower compared to censuses, it takes shorter time and in-depth data can be collected. However sampling has the problem of representation, and accuracy. These can be mitigated by using an appropriate sampling technique (see below) and increasing the sample size (the number of items selected into a sample.)

10.1 IMPORTANT TERMS

10.1.1 Population
A population is the collection of all people or items with the characteristics that one wishes to understand in an enquiry.

Example

For a survey collecting data on family size in a district. The populations are all the families in the district.

10.1.2 Population size
The numbers elements in the population. This number is often denoted by N.

10.1.3 Sample
A sample is a collection of items selected from the population. A sample is therefore a part of the population.
10.1.4 Sampling frame
This is an itemized list of the items from which a sample is to be drawn. The term sampling frame is different from population because it refers to a situation where the elements of a population can be and have been individually identified for purposes of sampling. With a sampling frame it is possible to state exactly the proportion of the population/frame which has been sampled. Not all populations render themselves to the construction of a sampling frame.

10.1.5 Sample size
This is the number of items in a sample. Sample size is often denoted by \( n \). For example, given a population has 250 element and 25 items are selected into a sample, then the sample size \( n \) is 25.

10.2 SAMPLING TECHNIQUES

10.2.1 Major groupings of sampling techniques.

a) Random or scientific techniques:
   These are techniques where sample elements have a defined chance of being selected into the sample.

b) Non Random or non scientific:
   Sample items do not have a defined probability of being selected.

10.2.2 Sampling techniques.

A. Overview
The most common sampling designs are simple random sampling, stratified random sampling, and multistage random sampling.

B. The simple random sampling (SRS)

**Simple random sampling** is a random (scientific) sampling technique where each individual is chosen entirely by chance and each member of the population has an equal chance of being included in the sample.

A simple illustration of simple random sampling is what happens with raffle ticket draws. Several tickets are crumpled thrown into a drum and reshuffled. A person is blind folded and asked to pick a ticket or tickets. The tickets so picked are picked purely at random.

In the case of sample surveys random numbers are used to achieve this randomness. Random numbers are number often generated by computer and they are such that one cannot predict any particular number from the set. They do not have a pattern nor is it possible to fit a formula to them.

**Example 1:**

Given a list of people in the following table clearly identified and numbered from 01 to 20, select a simple random sample.
Table 10.1 Sampling frame

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>D</td>
<td>Magwera</td>
<td>11</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>L</td>
<td>Mphande</td>
<td>12</td>
<td>M</td>
</tr>
<tr>
<td>3</td>
<td>E</td>
<td>Muyandira</td>
<td>13</td>
<td>I</td>
</tr>
<tr>
<td>4</td>
<td>W</td>
<td>Mhango</td>
<td>14</td>
<td>M</td>
</tr>
<tr>
<td>5</td>
<td>P</td>
<td>Chisambo</td>
<td>15</td>
<td>E</td>
</tr>
<tr>
<td>6</td>
<td>T</td>
<td>Tilenga</td>
<td>16</td>
<td>M</td>
</tr>
<tr>
<td>7</td>
<td>H</td>
<td>Ali</td>
<td>17</td>
<td>J</td>
</tr>
<tr>
<td>8</td>
<td>A</td>
<td>Lwanda</td>
<td>18</td>
<td>P</td>
</tr>
<tr>
<td>9</td>
<td>J</td>
<td>Sikwese</td>
<td>19</td>
<td>E</td>
</tr>
<tr>
<td>10</td>
<td>L</td>
<td>Ndawambe</td>
<td>20</td>
<td>T</td>
</tr>
</tbody>
</table>

Solution

Since the people in the table are clearly numbered, the list is the sampling frame. The desired sample size is 5, therefore the first 5, two-digit random numbers are selected from a random number table as shown.

Table 10.2 Selected random numbers

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>21</td>
<td>6</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>22</td>
<td>38</td>
<td>55</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>22</td>
<td>6</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>92</td>
<td>19</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>59</td>
<td>56</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>96</td>
<td>17</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>54</td>
<td>8</td>
<td>83</td>
<td>75</td>
<td></td>
</tr>
<tr>
<td>59</td>
<td>50</td>
<td>89</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>69</td>
<td>52</td>
<td>63</td>
<td></td>
</tr>
<tr>
<td>67</td>
<td>77</td>
<td>32</td>
<td>81</td>
<td></td>
</tr>
</tbody>
</table>

From the random number table, going vertically, the following are the first 5 random numbers between 01 and 20: 17, 01, 13, 07, and 10

Thus the elements of the sampling frame selected into the sample are:

01 D Magwera*
07 H Ali*
10 L Ndawambe*
13 J Jemu*
17 J Nyirenda*

Simple random sampling has the following advantages and disadvantages:
Advantages
i. It minimizes bias
ii. Because of low bias the sample is usually representative.
iii. The method is generally accepted by layman as fair

Disadvantages
i. The sample is venerable to sampling error
ii. The elements selected may not reflect the population.
iii. SRS may be Cumbersome and tedious.
iv. SRS will always require a sampling frame.
v. There is chance that certain significant attributes of the population are over or under represented

C. Systematic sampling

Systematic sampling is a technique where sample items are selected at a regular interval from a random start. The interval at which the population elements are picked is called a sampling interval. That is sampling is carried out by picking say every 5th, 8th, etc item depending on the sampling interval determined.

The sampling interval is calculated as follows

Let  
k = the sampling interval
N = the population size
n = the sample size

Then  
k = N/n

Example 2:

Select a sample of 5 people from the list shown in 3.2.1 above using systematic sampling

Solution

N = 20,  n = 5  the sampling interval k = 20/5 = 4

In such a case, sampling proceeds by taking every 4th item fro a random start in the first 4 elements of the sampling frame (the in the first interval).

Using the same random number table the first number between 01 and 04 is 01. This means starting with item 01 selecting every 4th element from the population, we have:

01, 05, 09, 13, and 17

The names are:
Advantages

i. The method is practical and easy to carry out on the ground.
ii. It does not require a strict sampling frame. It only requires the arrangement of the elements in the population.

Disadvantage

i. Systematic sampling can produce biased results if the population has periodicity in the elements. That is the arrangement forms a pattern and the sampling interval resonates with that pattern then only elements of a similar type will be selected.

ii. The method is not truly random since (once a random starting point has been selected) all subjects are pre-determined.

D. Stratified sampling

This is a sampling technique where the population is first divided into categories according to pre-existing characteristics in the population. The categories are called “strata” (Single stratum).

A sample is then drawn from each stratum and results combined later. For example, a survey on industrial turn over industries can be stratified as follows:

<table>
<thead>
<tr>
<th>Stratum</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Large industries</td>
</tr>
<tr>
<td>2</td>
<td>medium</td>
</tr>
<tr>
<td>3</td>
<td>small</td>
</tr>
</tbody>
</table>

Stratified sampling techniques are generally used when the population is heterogeneous, or dissimilar, and certain homogeneous, or similar, sub-populations can be isolated (strata). Simple random sampling is most appropriate when the entire population from which the sample is taken is homogeneous. Any appropriate sampling technique can then be used in the individual stratum:

The general procedure for taking a stratified sample is:

i. Stratify the population, defining a number of separate partitions.
ii. Calculate the proportion of the population lying on each partition.
iii. Split the total sample size up into the above proportions.
iv. Take a separate sample (normally simple random) from each partition, using the sample sizes as defined in (iii).
v. Combine the results to obtain the required stratified sample.
Advantages

i. It enables researchers to draw inferences about specific sub groups. That may be lost in a more general random sample.
ii. It leads to better statistical estimates.
iii. Stratified sampling makes data more readily available for pre existing subgroup within a population.
iv. If strata are treated independently different sampling technique can be used depending on the data required and a situation on the ground.

Disadvantages

i. The technique requires selection of a relevant clarification criteria.
ii. It can be expensive due to extra time and manpower required for the organisation and implementation of the sample.
iii. It requires an extensive sampling frame.
iv. Strata levels of importance can only be selected subjectively.
v. It is not useful when there are no homogeneous subjects.

E. Multistage sampling

A multistage sample is constructed by taking a series of samples in stages starting with wider area or definition of units to narrower and more specific units. The method is used where a population is spread over a relatively wide geographical area and SRS would require travelling to all parts of the area.

The method is as follows:

i. Split the area up into a number of regions
ii. Randomly select a small number of regions
iii. Confine sub-samples to these regions alone with size of each sub-sample proportional to the size of the area.
iv. The above procedure can be repeated for sub-regions within regions.

Once the final regions have been selected the final sampling technique could be random or systematic depending on the existence or otherwise of a sampling frame.

Example 3:

A researcher wants to estimate the proportion of household above a poverty line in terms of income in rural Malawi. Help the researcher design a sample.

Solution:

Taking the country as a whole the sample design would as follows:

Stage 1  Pick districts using SRS
Stage 2  From each district select a number of Traditional Areas
Stage 3  From each T/A select a number of Village headmen
Stage 4 From each Village headman select the required number of households.

The number of units (districts, traditional authorities, village headmen and households) being sampled at each stage depends on the accuracy required. Larger samples tend to give better accuracy.

Advantages

i. Cost of canvassing is low
ii. Data can be collected with relative speed due to ease of organisation the technique can give.
iii. It has convenience of finding the survey sample
iv. Normally more accurate than cluster sampling for the same size sample

Disadvantages

i. Is not as accurate as SRS if the sample is the same size
ii. More testing is difficult to do
iii. Possible bias if a very small number of regions is selected.
iv. No member of the population in any other region can be selected.

F. Cluster sampling

Cluster sampling is used where it is difficult or impossible to isolate the target responding units (elements of the population from which data is collected). In such cases it is more cost-effective to select respondents in groups ('clusters'). Grouping or clustering is often based on geography, or by time periods. Data is then collected from all respondents in a selected cluster.

Example 4

An NGO has read in the papers that very few people use mosquito nets in Ndirande-Zambia slam township. To confirm this, the NGO would like to collect first hand data necessary for intervention. However the Director is at pains as to what sampling method can be used because the dwelling units are not numbered and they are not arranged in proper rows. He has turned to you for help in the matter. Suggest a sampling method.

Solution

The proper sampling technique is cluster sampling. Much as the dwelling units are not numbered or arranged in rows, there are small access roads which form a network. The roads form boundaries of groups or clusters of houses. These clusters can be identified and an SRS can be picked and data collected from every household in a selected cluster

Example 5:

A manager in Peoples Trading Centre would like to find out from shoppers whether or not they like the new arrangement of goods. Help him draw a sample of the shoppers.
Solution
It is difficult to identify all hoppers individually before hand and therefore a sampling frame cannot be constructed. The best way is to cluster shoppers on time. Possible clusters are:

Treating shoppers that enter the shop starting at 08:00 can be grouped in periods of 15 minutes up to say close of the shop at 20:00. A systematic sample of these groupings can be made. One possible such sample is to pick the groupings every two hours as follows

08:00 – 08:15
10:00 – 10:15
12:00 – 12:15
14:00 – 14:15
16:00 – 16:15
18:00 – 18:15

Note that a SRS sample can also be drawn since the time clusters can be uniquely identified.

Advantages
i. Clustering can reduce travel and administrative costs. An interviewer can make a single trip to visit several responding units in one block, rather than having to drive to a different block for each unit.
ii. It also means that one does not need a sampling frame listing all elements in the target population. Instead, clusters can be chosen from a cluster-level frame, with an element-level frame created only for the selected clusters.

Disadvantages
i. Cluster sampling requires a larger sample than SRS to achieve the same level of accuracy because of the variability among clusters.
ii. The fact that the method is not random sampling then the selection bias could be significant.

G. Quota sampling
Quota sampling is one of the non-scientific techniques of sampling. The researcher decides on the number of sampling units (i.e sample size) according to the required coverage (purely from judgement). The data collector then collects data from any responding units in the population up to the number arrived at earlier.

Advantages
i. The technique is simple and convenient
ii. There is no non-response
iii. The cost of data collection is very low,
iv. Quota sampling does not require a sampling frame.

Disadvantage

i. The technique produces very biased results because of the human interference in the sample selection.
ii. Severe interviewer bias can be introduced into the survey by inexperienced or untrained interviewers, since all the data collection and recording rests with them

CHAPTER SUMMARY

In this chapter you have learned that:

- Census is a survey which examines every member of the population
- A sample is a relatively small subset of the population with advantage over census that cost, time and other resources are much lower
- A sampling frame is a structure which lists or identifies the members of the a population.
- Simple random sampling is a technique which ensures that each and every of the population has an equal chance of being chosen for the sample.
- Stratified random sampling ensures that every significant group in the population is represented in proportion in the sample.
- Multi-stage (quasi-random) sampling is normally used in homogeneous population spread over a wide area.
- Stratified (quasi-random) sampling involves selecting a random starting point and then sampling every nth member of the population.
- Cluster (quasi-random) sampling exhaustive sampling from a few well chosen areas
- Quota (quasi-random) sampling normally involves teams in of interviewers who obtain information from a set quota of people.

END OF CHAPTER EXERCISES

1 Define and give examples of the following terms
a) Population
b) Sampling frame
c) Sampling unit
d) Random numbers
e) Responding unit

2 Compare and contrast probability and non-probability sampling techniques.

3 Systematic sampling may produce a biased sample. Discuss
4 How can you collect a sample using the following sampling techniques

Simple random sampling
Cluster sampling
Multistage sampling
Stratified sampling

5 Suggest a suitable sampling technique in each of the following situations. Describe the technique and state reason for its suitability.

(i) In an audit you wish to form an opinion from a number of invoices received throughout the year.

(ii) You are trying to estimate maize field over a 5 hectare field.

(iii) You want to gather opinions from spectators on the strength of two soccer teams just before a match.
CHAPTER 11  DATA PRESENTATION

Learning Objectives:

By the end of this chapter, you should be able to:

i. explain the need to represent statistical information
ii. understand the various techniques for presenting data
iii. select appropriate data presentation techniques for specific types of data
iv. construct diagrammatic displays of data
v. interpret graphical displays of data

11.0  INTRODUCTION

Before looking at all different techniques for presenting data, it is necessary to consider the purpose for which the data was collected. For instance, the data you collected might have been wanted for your company’s annual report. A straightforward list of all data values could be presented but, particularly if there were a lot of items, this would not be very helpful and even very boring.

Data presentation therefore simplifies large amounts of data, shows key facts and patterns, and display data in an interesting and easily understandable way. This generally involves sorting and grouping, illustration and summary statistics. In this chapter, we will focus on sorting, grouping and illustrating data using tables, charts and graphs.

Common data presentation techniques include:

- Frequency table/distribution
- Pictogram
- Bar charts: Simple, multiple (compound) and component (stacked) bar charts
- Pie chart
- Histogram
- Frequency polygon
- Cumulative frequency graph (ogive)
- Lorenz curve
- Z-chart

11.2  FREQUENCY DISTRIBUTION

Definition

A frequency is the number of observations or items or data values that belong to each category or class of data. For qualitative or discrete quantitative data, a frequency is simply a record of how many of each
type were present. Consequently, a **Frequency Distribution** can be said to be a grouping of data into mutually exclusive classes showing the number of observations or items in each class.

The main aim of a frequency distribution is to summarise data in a logical manner that enables an overall perspective of data to be obtained quickly. A frequency distribution can be represented in form of a table, a graph or a formula. In this module, however, we shall only look at frequency distributions that take the forms of a table and/or graph.

Frequency distributions can be classified into simple and grouped distributions. While simple distributions involve qualitative and discrete quantitative data, grouped distributions involve continuous quantitative data.

Table 11.1 is a frequency distribution showing the frequency with which some ports of exit were used by departing visitors in 2009.

**Table 11.1: Number of visitors departing Malawi by port of exit, 2009**

<table>
<thead>
<tr>
<th>Port of exit</th>
<th>Number of visitors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chileka</td>
<td>76,308</td>
</tr>
<tr>
<td>Chiponde</td>
<td>52,836</td>
</tr>
<tr>
<td>Chitipa/Chisenga</td>
<td>6,816</td>
</tr>
<tr>
<td>Dedza</td>
<td>10,916</td>
</tr>
<tr>
<td>Kaporo/Songwe</td>
<td>90,350</td>
</tr>
<tr>
<td>Lilongwe (KIA)</td>
<td>174,652</td>
</tr>
<tr>
<td>Mchinji/Chimaliro</td>
<td>82,304</td>
</tr>
<tr>
<td>Muloza</td>
<td>41,203</td>
</tr>
<tr>
<td>Mwanza</td>
<td>141,956</td>
</tr>
<tr>
<td>Nayuchi</td>
<td>14,759</td>
</tr>
<tr>
<td>Nsanje/Marka</td>
<td>17,322</td>
</tr>
<tr>
<td>Other</td>
<td>45,609</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>755,031</strong></td>
</tr>
</tbody>
</table>

*Source: 2009 Tourism Report - NSO*

From the table above, it is easy to see that visitors departing Malawi frequently used Kamuzu International Airport (KIA) as their port of exit in 2009. A total of 174,652 visitors exited Malawi through Lilongwe (KIA). The simple frequency distribution in Table 10.1 can be transformed into a complex distribution by among other things including the reasons for visiting and the gender of the visitor as shown in Table 11.2.
Table 11.2: Number of visitors departing Malawi by port of exit, gender and reason for visit

<table>
<thead>
<tr>
<th>Port of exit</th>
<th>Holiday/vacation</th>
<th>Work/business</th>
<th>Visit friends/relatives</th>
<th>Conference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Female</td>
<td>Male</td>
<td>Female</td>
<td>Male</td>
</tr>
<tr>
<td>Chileka</td>
<td>7226</td>
<td>8353</td>
<td>9378</td>
<td>42638</td>
</tr>
<tr>
<td>Chiponde</td>
<td>10250</td>
<td>28391</td>
<td>2511</td>
<td>8968</td>
</tr>
<tr>
<td>Chitipa/Chisenga</td>
<td>512</td>
<td>769</td>
<td>256</td>
<td>4254</td>
</tr>
<tr>
<td>Dedza</td>
<td>4049</td>
<td>5688</td>
<td>154</td>
<td>307</td>
</tr>
<tr>
<td>Kaporo/Songwe</td>
<td>7533</td>
<td>17065</td>
<td>3075</td>
<td>35772</td>
</tr>
<tr>
<td>Lilongwe (KIA)</td>
<td>17117</td>
<td>28084</td>
<td>17527</td>
<td>89837</td>
</tr>
<tr>
<td>Mchinji/Chimaliro</td>
<td>5842</td>
<td>11018</td>
<td>5074</td>
<td>41716</td>
</tr>
<tr>
<td>Muloza</td>
<td>8251</td>
<td>17527</td>
<td>1845</td>
<td>9327</td>
</tr>
<tr>
<td>Mwanza</td>
<td>16758</td>
<td>23523</td>
<td>8866</td>
<td>62010</td>
</tr>
<tr>
<td>Nayuchi</td>
<td>2716</td>
<td>6047</td>
<td>1230</td>
<td>4561</td>
</tr>
<tr>
<td>Nsanje/Marka</td>
<td>564</td>
<td>2716</td>
<td>1947</td>
<td>9532</td>
</tr>
<tr>
<td>Other</td>
<td>2716</td>
<td>12813</td>
<td>3382</td>
<td>16963</td>
</tr>
<tr>
<td>Total</td>
<td>83534</td>
<td>161994</td>
<td>55245</td>
<td>325884</td>
</tr>
</tbody>
</table>

Source: 2009 Tourism Report – NSO

Let us now construct our own frequency distribution.

Example 1
A BMS class at an Accountancy College has 42 students. Of these, 19 have a Certificate in Financial Accounting (CIFA) background. There are 4 male students with CIFA background while 15 female students have the CIFA background. In total there are 19 female and 23 male students. Present the data on a table.

Solution:

<table>
<thead>
<tr>
<th>Number of students in a BMS class</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>CIFA background</td>
</tr>
<tr>
<td>Male</td>
</tr>
<tr>
<td>Female</td>
</tr>
<tr>
<td>Total</td>
</tr>
<tr>
<td>Other</td>
</tr>
<tr>
<td>Male</td>
</tr>
<tr>
<td>Female</td>
</tr>
<tr>
<td>Total</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>

Constructing a Frequency Distribution for Quantitative Data

We can construct a frequency distribution from an array or table of numbers (data values) using the following steps as a guide:
Step 1: Decide on the number of classes, \( k \)
We are supposed to use just enough classes or groups of data values so that we reveal the shape of the distribution. The number of classes depends on the number of observations (\( n \)) in the data set. Traditionally, the number of classes must vary from 5 to 20 i.e. \( 5 \leq k \leq 20 \)
A quick hint to decide on the number of classes is the “2 raised to the \( k \) rule”. The rule states that we must select the smallest \( k \) number of classes such that \( 2^k \geq n \)

Step 2: Determine the class width or interval
The class interval or width must, in general, be the same for all the classes. The class width is expressed as:

\[
i \geq \frac{x_{\text{max}} - x_{\text{min}}}{k}
\]

where \( i \) is the class interval/width, \( x_{\text{max}} \) is the largest or highest data value, \( x_{\text{min}} \) is the smallest or lowest data value in the raw data and \( k \) is the number of classes.

Step 3: Determine the individual class boundaries
Set the class boundaries such that each data value can be put into only one category. This means we must avoid overlapping class boundaries. For example, the classes 10 – 20 and 15 – 25 should not be used in the same distribution because they are overlapping.

In order to get the classes right, you need to set up the correct lower boundary of the first or lowest class. The lower boundary for the first class can be the lowest data value (\( x_{\text{min}} \)) in the data set or any number slightly lower than the lowest value. For instance, if the lowest data value is 21 then the lower boundary of the first class can be 21 or any number slightly lower that 21 e.g. 20. The rest of the class boundaries are then determined based on this lower boundary.

Note that as you decide on the classes they will be adjustments. Ensure that figures used for class boundaries are numbers which are easy to work with e.g. numbers divisible by 2, 5 or 10.

Step 4: Tally the data values
Use tallies to allocate counts of data values for each class.

Step 5: Count the number of items
Count the tally marks in each class to obtain the class frequencies. The frequency distribution/table can then be rewritten so that it is presented without the tally marks.

Example 2
A fisherman using a line and rod recorded his catch per day for 50 days and his records are as following.
Construct a frequency distribution for these data.

**Solution**

**Step 1: Number of classes**

Since there are \( n = 50 \) data values, we need to have \( k \) classes such that \( 2^k \geq 50 \). For \( k = 5 \), \( 2^5 = 32 \) which is less than 50. Consequently, 5 classes are not enough for this data set. If we set \( k = 6 \), then \( 2^6 = 64 \), which is greater than 50. So we recommend \( k = 6 \) classes for the data set.

**Step 2: Class width/interval**

Since the lowest catch/day is 5 and the highest catch/day is 46, and that we need 6 classes, the class width should be at least \( i = \frac{46 - 5}{6} = 6.8 \). Usually, such a class width would be rounded up to some convenient number such as the next integer, a multiple of 5, 10 or 100. In our case, we round 6.8 to 7.

**Step 3: Individual class boundaries**

Since the lowest catch/day is 5, the lowest boundary for the first class can be 5 or any number slightly smaller than 5 such as 4 and 3. We are of the view that ‘5’ is much easier to work with, so we set the lower boundary of the first class at 5. Consequently, here are the 6 classes for this data set: 5 – 12, 12 – 19, 19 – 26, 26 – 33, 33 – 40, 40 – 47

**Step 4: Tallying**

**Table 11.4 Fish catch tallies**

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>Tallies</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>/////</td>
</tr>
<tr>
<td>12 – 19</td>
<td>//////////</td>
</tr>
<tr>
<td>19 – 26</td>
<td>/////////////</td>
</tr>
<tr>
<td>26 – 33</td>
<td>###</td>
</tr>
<tr>
<td>33 – 40</td>
<td>///</td>
</tr>
<tr>
<td>40 – 47</td>
<td>//</td>
</tr>
</tbody>
</table>

**Note:** Each class includes the lower boundary but not the upper boundary. For example, the data value 26 is in the class 26 – 33 and not in the class 19 – 26.

**Step 5: Counting the number of items or tallies to obtain the class frequencies**
Table 11.5  Fish catch frequency table

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>5</td>
</tr>
<tr>
<td>12 – 19</td>
<td>16</td>
</tr>
<tr>
<td>19 – 26</td>
<td>14</td>
</tr>
<tr>
<td>26 – 33</td>
<td>9</td>
</tr>
<tr>
<td>33 – 40</td>
<td>4</td>
</tr>
<tr>
<td>40 – 47</td>
<td>2</td>
</tr>
</tbody>
</table>

11.2.1 General Forms of Frequency Distributions

The following examples show typical forms of frequency distributions. Each example tries to depict a special feature that one may find in a frequency distribution.

i) A simple frequency distribution shows singles values and their frequencies (counts), and it is useful when summarizing simple discrete data over a limited range.

Table 11.6  Simple frequency distribution

<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>17</td>
<td>8</td>
</tr>
<tr>
<td>21</td>
<td>6</td>
</tr>
<tr>
<td>25</td>
<td>3</td>
</tr>
<tr>
<td>30</td>
<td>5</td>
</tr>
</tbody>
</table>

ii) The following frequency distribution shows the number of orders that a company received per week over a 40-weeks period. Notice that the classes are continuous and of equal width.

Table 11.7  Orders received over 40 weeks

<table>
<thead>
<tr>
<th>Number of order</th>
<th>Number of weeks (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 but less than 10</td>
<td>1</td>
</tr>
<tr>
<td>10 but less than 15</td>
<td>5</td>
</tr>
<tr>
<td>15 but less than 20</td>
<td>7</td>
</tr>
<tr>
<td>20 but less than 25</td>
<td>10</td>
</tr>
<tr>
<td>25 but less than 30</td>
<td>7</td>
</tr>
<tr>
<td>30 but less than 35</td>
<td>4</td>
</tr>
<tr>
<td>35 but less than 40</td>
<td>3</td>
</tr>
</tbody>
</table>
iii) Here the classes are not continuous. There are gaps for example between 14 and 15, between 19 and 20, etc. These are known as class limits and not boundaries. This is typical for a frequency distribution where data is discrete.

Table 11.8 Students Age distribution

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 – 14</td>
<td>50</td>
</tr>
<tr>
<td>15 – 19</td>
<td>80</td>
</tr>
<tr>
<td>20 – 24</td>
<td>45</td>
</tr>
<tr>
<td>25 – 29</td>
<td>12</td>
</tr>
<tr>
<td>30 – 34</td>
<td>10</td>
</tr>
</tbody>
</table>

We can transform the class limits to boundaries by simply averaging the adjacent limits. The class boundary between the classes 10 – 14 and 15 – 19 would be the average of 14 and 15, i.e. \( \frac{14 + 15}{2} = 14.5 \), and the boundary between the classes 15 – 19 and 20 – 24 would be the average of 19 and 20 i.e. \( \frac{19 + 20}{2} = 20.5 \). We can therefore represent the distribution above using class boundaries as shown below:

Table 11.9 Students age distribution

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.5 – 14.5</td>
<td>50</td>
</tr>
<tr>
<td>14.5 – 19.5</td>
<td>80</td>
</tr>
<tr>
<td>19.5 – 24.5</td>
<td>45</td>
</tr>
<tr>
<td>24.5 – 29.5</td>
<td>12</td>
</tr>
<tr>
<td>29.5 – 34.5</td>
<td>10</td>
</tr>
</tbody>
</table>

Note: The classes are now continuous.

iv) The classes in the following distribution do not have equal width. In this case, the width will depend on the requirements of the individual presenting the data.

Table 11.10 Frequency distribution: unequal class width

<table>
<thead>
<tr>
<th>Days</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 – 10</td>
<td>8</td>
</tr>
<tr>
<td>10 – 15</td>
<td>12</td>
</tr>
<tr>
<td>15 – 20</td>
<td>10</td>
</tr>
<tr>
<td>20 – 22</td>
<td>16</td>
</tr>
<tr>
<td>22 – 25</td>
<td>11</td>
</tr>
</tbody>
</table>
v) This distribution has two open ended classes: under 100 and 250 and over. Open ended classes are used when data at the ends of the distribution are too widely and sparsely spread.

Table 11.11  Frequency distribution with open ended classes.

<table>
<thead>
<tr>
<th>Income (in K'000)</th>
<th>Number of employees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under 100</td>
<td>116</td>
</tr>
<tr>
<td>100 and under 120</td>
<td>89</td>
</tr>
<tr>
<td>120 and under 150</td>
<td>184</td>
</tr>
<tr>
<td>150 and under 170</td>
<td>142</td>
</tr>
<tr>
<td>170 and under 200</td>
<td>159</td>
</tr>
<tr>
<td>200 and under 250</td>
<td>124</td>
</tr>
<tr>
<td>250 and over</td>
<td>124</td>
</tr>
</tbody>
</table>

11.3  CLASS INTERVALS/WIDTH AND MIDPOINTS

In this module, the terms ‘class interval/width’ and ‘class midpoints’ are used frequently. You need, therefore, to have a clear understand of the two. The class midpoint or class mark is halfway the between the lower and upper boundaries (or limits) of the same class. It is computed by adding the lower and upper boundaries and dividing the results by 2. Refer to the fisherman’s example, the midpoint for first class (5 – 12) with 5 as the lower boundary and 12 as its upper boundary is 8.5 found by \((5+12)/2\).

The rest of the midpoints are:

Table 11.12  Frequency distribution: class mid-point calculated

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>f</th>
<th>Midpoint, x</th>
<th>Midpoint found by</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>5</td>
<td>8.5</td>
<td>((5+12)/2)</td>
</tr>
<tr>
<td>12 – 19</td>
<td>16</td>
<td>15.5</td>
<td>((12+19)/2)</td>
</tr>
<tr>
<td>19 – 26</td>
<td>14</td>
<td>22.5</td>
<td>((19+26)/2)</td>
</tr>
<tr>
<td>26 – 33</td>
<td>9</td>
<td>29.5</td>
<td>((26+33)/2)</td>
</tr>
<tr>
<td>33 – 40</td>
<td>4</td>
<td>36.5</td>
<td>((33+40)/2)</td>
</tr>
<tr>
<td>40 – 47</td>
<td>2</td>
<td>43.5</td>
<td>((40+47)/2)</td>
</tr>
</tbody>
</table>

To determine the class interval/width, subtract the lower class boundary from the upper class boundary. We can also determine the class interval/width by finding the difference between consecutive midpoints. The class interval for the fisherman’s example was determined as 7, which we find by subtracting the lower boundary of the first class, 5, from the upper boundary of the same class, 12 i.e. 12 – 5 = 5. Alternatively, since the midpoint of the first class is 8.5 and the midpoint of the second class is 15.5, then the class width is \(15.5 – 8.5 = 7\).
11.4 RELATIVE AND CUMULATIVE FREQUENCIES

It may be desirable to convert frequencies to relative and/or cumulative frequencies. A relative frequency is the ratio of the class frequency to the total number of observations. It shows the fraction (or percentage) of the total number of data values in each class.

Mathematically, a relative frequency of each class/data value is defined as:

$$\text{Relative frequency} = \frac{\text{Frequency of a class or data value}}{\text{Sum of the frequencies}} = \frac{f}{\sum f}$$

In our fisherman example, we may want to show the proportion of days on which specific intervals of fish catch were recorded as shown below:

Table 11.13 Relative frequency distribution

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>Frequency</th>
<th>Relative frequency</th>
<th>Found by</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>5</td>
<td>0.10</td>
<td>5 / 50</td>
</tr>
<tr>
<td>12 – 19</td>
<td>16</td>
<td>0.32</td>
<td>16 / 50</td>
</tr>
<tr>
<td>19 – 26</td>
<td>14</td>
<td>0.28</td>
<td>14 / 50</td>
</tr>
<tr>
<td>26 – 33</td>
<td>9</td>
<td>0.18</td>
<td>9 / 50</td>
</tr>
<tr>
<td>33 – 40</td>
<td>4</td>
<td>0.08</td>
<td>4 / 50</td>
</tr>
<tr>
<td>40 – 47</td>
<td>2</td>
<td>0.04</td>
<td>2 / 50</td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Note: We can convert the relative frequencies to percentages simply by multiplying them by 100 (%).

Definition: Less-than and more-than cumulative frequencies

A less-than cumulative frequency is the number of data values/observations that are less than a specific data value fall or below the upper boundary of a specific class in case of grouped frequency distribution. The less-than cumulative frequency is the default cumulative frequency. However, a more-than cumulative frequency is the number of data values/observations that are equal to or greater than a specific data value or the lower boundary of specific class in case of grouped frequency distribution.

Referring to our fisherman example, we may want to show the number of days on which he recorded less than 19 fishes. In this case we would simply add 5 and 16 to give us 21 days. And for less than 26 fishes, we add 5, 16 and 14 to get 35 days. Furthermore, if we wanted the number of days on which he recorded either equal to or more that 26 fishes, we would simply add 9, 4 and 2 to get 15 days. The cumulative frequencies for the remaining classes are shown in the table below:
Table 11.14 “Less-than” cumulative frequency distribution.

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>Frequency f</th>
<th>“Less-than” cumulative, F</th>
<th>Found by</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>12 – 19</td>
<td>16</td>
<td>21</td>
<td>5+16</td>
</tr>
<tr>
<td>19 – 26</td>
<td>14</td>
<td>35</td>
<td>5+16+14</td>
</tr>
<tr>
<td>26 – 33</td>
<td>9</td>
<td>44</td>
<td>5+16+14+9</td>
</tr>
<tr>
<td>33 – 40</td>
<td>4</td>
<td>48</td>
<td>5+16+14+9+4</td>
</tr>
<tr>
<td>40 – 47</td>
<td>2</td>
<td>50</td>
<td>5+16+14+9+4+2</td>
</tr>
</tbody>
</table>

Table 11.15 “More-than” cumulative frequency distribution

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>Frequency f</th>
<th>Less-than cumulative, F</th>
<th>Found by</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>5</td>
<td>50</td>
<td>2+4+9+14+16+5</td>
</tr>
<tr>
<td>12 – 19</td>
<td>16</td>
<td>45</td>
<td>2+4+9+14+16</td>
</tr>
<tr>
<td>19 – 26</td>
<td>14</td>
<td>29</td>
<td>2+4+9+14</td>
</tr>
<tr>
<td>26 – 33</td>
<td>9</td>
<td>15</td>
<td>2+4+9</td>
</tr>
<tr>
<td>33 – 40</td>
<td>4</td>
<td>6</td>
<td>2+4</td>
</tr>
<tr>
<td>40 – 47</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

11.5 ADVANTAGES AND DISADVANTAGES OF TABULAR PRESENTATION

Advantages of Tables
- Enhanced level of accuracy and precision
- Data can be presented in more than one dimension (more than one variable). For example in Table 10.2, we have the following variables presented: Port of exit (name), reason for visit and gender of the visitors.

Disadvantage of a Table
- Tables lack visual impression of the distribution
- The figures may be cumbersome (especially in large tables)

11.6 GRAPHICAL AND DIAGRAMMATIC PRESENTATION OF FREQUENCY DISTRIBUTION

Managers and other busy individuals often need a quick picture of the trend in variables of interest such as sales, expenditure, cost, revenue, and profit. These trends are usually easy to depict using charts and graphs.
11.6.1 Pictogram
A pictogram is a technique where pictures or symbols are used to represent data. For example, if 500 houses were built in Ndirande and 250 houses in Zolozolo, then using a scale of 1 picture of a house to 50 houses, we can represent the information as follows:

![Pictogram](image1)

**Figure 11.1 Pictogram**

Ndirande: 
Zolozolo: 
Key: = 50 houses

**Example 3**
A survey was carried out to find the number of school buses operating in the major towns in Malawi. The results are presented in the distribution below. Present the distribution using a pictogram.

<table>
<thead>
<tr>
<th>Town</th>
<th>Number of school buses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blantyre</td>
<td>200</td>
</tr>
<tr>
<td>Lilongwe</td>
<td>150</td>
</tr>
<tr>
<td>Zomba</td>
<td>50</td>
</tr>
<tr>
<td>Mzuzu</td>
<td>20</td>
</tr>
</tbody>
</table>

**Table 11.16 School buses**

**Solution:**

![Pictogram](image2)

Blantyre: 
Lilongwe: 
Zomba: 
Mzuzu: 

Key: = 50 school buses

**Advantage of a Pictogram**
- A pictogram provides a quick visual impression of the data (magnitude)
Disadvantages of a Pictogram
- It lacks precision specially interpreting fractions of the pictures/symbols used. For example, the immediate problem that comes out is how to express the 20 buses in Mzuzu in the above problem.
- The technique has limited dimensions in which data can be presented.

11.6.2 Bar Chart
A bar chart or bar graph is a chart with non-joining rectangular bars. Data is represented by the bars and the lengths or heights are proportional to the values or frequencies that they represent. The bars can be plotted vertically or horizontally.

There are three common forms of bar charts namely:
- The simple bar chart
- The multiple or compound bar chart
- The component or stacked bar chart

How to Construct a Bar Chart
1) Place the data categories on the x-axis (no scaling is required)
2) Place the frequencies/counts/percentages on y-axis and must be scaled accordingly. The height or length of each bar will show the frequency/percentage/count for each category.
3) The bars must be equal equally spaced and of equal width.

The Simple Bar Chart
A simple bar chart presents data only in two dimensions.

We typify the construction of a simple bar chart using the information in the table below on the levels of imports into Malawi in 2010.

Table 11.2: Malawi’s imports by Broad Economic Category (BEC) in 2010

<table>
<thead>
<tr>
<th>BEC description</th>
<th>CIF Value (MK‘Billion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food and beverages</td>
<td>26.0</td>
</tr>
<tr>
<td>Consumable goods</td>
<td>38.2</td>
</tr>
<tr>
<td>Fuel and lubricants</td>
<td>30.0</td>
</tr>
<tr>
<td>Capital goods</td>
<td>42.9</td>
</tr>
<tr>
<td>Passenger motor cars</td>
<td>7.1</td>
</tr>
<tr>
<td>Parts and accessories</td>
<td>8.2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>152.4</strong></td>
</tr>
</tbody>
</table>

*Source: Annual Statistics Trade Report 2010*
To construct the bar chart, we shall place BEC item descriptions on the x-axis and the CIF values in the y-axis. The bar graph is presented below.

Figure 11.3 Malawi Imports bar chart

With this chart it is easy to see which BEC registered the highest CIF value in 2010. We can also see that CIF value for consumable goods was more than four times the CIF value for passenger cars in 2010.

Multiple or Compound Bar Chart

A multiple bar chart is similar to a simple bar chart only that it does show more than one aspect of the data/variable. In a multiple bar chart, each bar represents a specific of the major category of a variable.

Example 4
The following data shows the annual external trade values for Malawi from 2005 to 2011.

Table 11.18: Annual external trade values for Malawi

<table>
<thead>
<tr>
<th>Year</th>
<th>Trade values (MK’Billion)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exports</td>
</tr>
<tr>
<td>2005</td>
<td>59.6</td>
</tr>
<tr>
<td>2006</td>
<td>93.8</td>
</tr>
<tr>
<td>2007</td>
<td>122.2</td>
</tr>
<tr>
<td>2008</td>
<td>123.6</td>
</tr>
<tr>
<td>2009</td>
<td>168.0</td>
</tr>
<tr>
<td>2010</td>
<td>159.4</td>
</tr>
<tr>
<td>2011</td>
<td>223.3</td>
</tr>
</tbody>
</table>
Present the trade values on a multiple or compound bar chart.

Solution
We proceed by constructing the bar chart clustered by year (i.e. years appear along the x-axis) and the charts is presented below.

Figure 11.4 Annual external trade values

One can tell from the chart that Malawi had been importing more than it had been export over the period 2005-2011.

It is also possible to construct a multiple bar chart clustered by trade type. In this case we shall put the trade in the x-axis. The bar chart would look like the one below:

Table 11.5 External trade values
We can tell from the chart that while exports had generally increased from 2005 to 2011, imports into Malawi had drastically increased over the same period.

**Component Bar Chart**

A component or stacked bar chart is more like the simple bar chart except that each bar, while representing the total (frequency/count/value), it is split up into the specific components of the variable category which are stacked on top of one another in the same order. For instance, if the bars are representing the number of students in a class, the bars would be split up show the number of male and female students in the class.

The following table shows holiday locations booked through a travel agent in Malawi.

<table>
<thead>
<tr>
<th>Holiday location</th>
<th>Annual bookings</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2001</td>
</tr>
<tr>
<td>Mangochi</td>
<td>385</td>
</tr>
<tr>
<td>Liwonde</td>
<td>186</td>
</tr>
<tr>
<td>Kasungu</td>
<td>140</td>
</tr>
<tr>
<td>Other</td>
<td>112</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>823</strong></td>
</tr>
</tbody>
</table>

We shall now proceed to construct a component bar chart for the data clustered by year and leave for your practice to construct a component bar chart clustered by holiday location.

The first step in constructing component bar chart is to find the totals. Luckily, the totals for each year (since we are clustering by year) are already provided in the table above. We place the years in the x-axis and number of bookings in the y-axis, scaled according. The component bar chart is presented below.
Note that for each year the four segments of the bar are stacked together to give the total number of bookings for each particular year. For 2001, the bars for Other (112), Kasungu (140) and Liwonde (186) are stacked onto the Mangochi bar (385) resulting into a combined bar for 2001 whose height/length is 823.

Quite often component bar charts are presented in terms of percentages. Unlike the absolute component bar chart we just constructed which requires the use of raw data frequencies, to construct a percentage bar chart we need to express the composition for each year as percentage of the total for that year. We explain the process of constructing a percentage component bar chart using the information in Table 11.5, which shows number of holiday bookings from 2001 to 2005.

The first step in constructing a percentage component chart is to express the components as a percentage of the total (for each year since we are clustering by year). For example, the Mangochi bookings are \(\frac{385}{823} \times 100 = 46.7 \approx 47\%\) of the 2001 total. In 2005, Kasungu bookings are \(\frac{184}{777} \times 100 = 23.7 \approx 24\%\) of the total. The rest of the percentages are given in the following table.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mangochi</td>
<td>385</td>
<td>47</td>
<td>350</td>
<td>47</td>
<td>326</td>
<td>37</td>
<td>341</td>
<td>39</td>
<td>286</td>
<td>37</td>
</tr>
<tr>
<td>Liwonde</td>
<td>186</td>
<td>23</td>
<td>178</td>
<td>24</td>
<td>224</td>
<td>25</td>
<td>212</td>
<td>24</td>
<td>195</td>
<td>25</td>
</tr>
<tr>
<td>Kasungu</td>
<td>140</td>
<td>17</td>
<td>156</td>
<td>21</td>
<td>187</td>
<td>21</td>
<td>188</td>
<td>21</td>
<td>184</td>
<td>24</td>
</tr>
<tr>
<td>Other</td>
<td>112</td>
<td>14</td>
<td>65</td>
<td>9</td>
<td>156</td>
<td>17</td>
<td>143</td>
<td>16</td>
<td>112</td>
<td>14</td>
</tr>
<tr>
<td>Total</td>
<td>823</td>
<td>100</td>
<td>749</td>
<td>100</td>
<td>893</td>
<td>100</td>
<td>884</td>
<td>100</td>
<td>777</td>
<td>100</td>
</tr>
</tbody>
</table>
We then construct a component bar chart just as before but this time around we will have percentages in the y-axis instead of raw number of bookings.

Table 11.7  Annual holiday bookings,

The chart shows that where as Mangochi had above 40% in 2001 and 2002 of the total number of bookings, its number of bookings dropped in percentage terms while that of Kasungu and Liwonde gained prominence in the subsequent years.

11.6.3 Pie Chart
A pie chart is normally useful for illustrating nominal level variables or data. In a pie chart data is presented on sectors of a circle. The size of a particular sector is proportional to the magnitude of a data item or the frequency of a data class. The proportionality is determined by assigning an appropriate angle at the centre of the circle.

How to construct a Pie Chart
Step 1: Find angles
Convert the data frequencies/values to angular measurements. This is done by dividing each frequency/value by their sum and then multiplying by $\frac{q}{360}$

Step 2: Draw a circle (pie)
Draw a circle of reasonable radius and divide it into segments/sectors proportional to the frequencies (or counts) as measured by the angles. It is expected that the sum of the sectors or angular measurements will be $\frac{q}{360}$ (or 100%)

Step 3: Shade the sectors or segments different and these must be explained in a “legend” or “key”

We reproduce the 2010 Malawi imports data from Table 11.4. Our intention is to construct a pie chart for the data.
Table 11.21  Malawi Exports

<table>
<thead>
<tr>
<th>BEC description</th>
<th>CIF Value (MK’Billion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food and beverages</td>
<td>26.0</td>
</tr>
<tr>
<td>Consumable goods</td>
<td>38.2</td>
</tr>
<tr>
<td>Fuel and lubricants</td>
<td>30.0</td>
</tr>
<tr>
<td>Capital goods</td>
<td>42.9</td>
</tr>
<tr>
<td>Passenger motor cars</td>
<td>7.1</td>
</tr>
<tr>
<td>Parts and accessories</td>
<td>8.2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>152.4</strong></td>
</tr>
</tbody>
</table>

The first step is to convert the CIF values to angles.

Table 11.22  Import values and angles of sectors of a pie chart

<table>
<thead>
<tr>
<th>BEC description</th>
<th>Value</th>
<th>Angle at the centre</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food and beverages</td>
<td>26.0</td>
<td>(\frac{26}{152.4} \times 360^\circ \approx 61.4^\circ)</td>
</tr>
<tr>
<td>Consumable goods</td>
<td>38.2</td>
<td>(\frac{38.2}{152.4} \times 360^\circ \approx 90.2^\circ)</td>
</tr>
<tr>
<td>Fuel and lubricants</td>
<td>30.0</td>
<td>(\frac{30}{152.4} \times 360^\circ \approx 70.9^\circ)</td>
</tr>
<tr>
<td>Capital goods</td>
<td>42.9</td>
<td>(\frac{42.9}{152.4} \times 360^\circ \approx 101.3^\circ)</td>
</tr>
<tr>
<td>Passenger motor cars</td>
<td>7.1</td>
<td>(\frac{7.1}{152.4} \times 360^\circ \approx 16.8^\circ)</td>
</tr>
<tr>
<td>Parts and accessories</td>
<td>8.2</td>
<td>(\frac{8.2}{152.4} \times 360^\circ \approx 19.4^\circ)</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>152.4</strong></td>
<td><strong>360^\circ</strong></td>
</tr>
</tbody>
</table>

The second step involves using a compass to draw a circle and a protractor to measure and assign angles at the centre of the circle. And finally the in third step you shade/colour the pie chart sectors. The final pie chart is as follows:
Because the area of the pie represents the relative share of each import item, we can easily compare them. From the pie chart, it is clear that capital goods accounted for the largest proportion of imports for Malawi in 2010. ‘Passenger car’ category was the least at 5% of the five broad economic categories in terms of their CIF values.

**Advantages:**

i. A pie chart can give a visual impression of the comparative sizes of the components
ii. A pie chart is relatively easy to understand
iii. Much as it is meant to provide a comparative picture of the components of the data, a pie chart can have actual data and/or percentages embedded in the diagram to show magnitudes

**Disadvantages:**

i. Lack of accuracy in general if figures are not embedded
ii. Limited dimensions in which data can be presented since data can only be presented in one dimension
iii. Need to calculate the angles, drawn the circle and draw sectors

**11.6.4 Histogram**

A histogram is one of the most common ways of presenting grouped frequency distribution, pictorially. A histogram is very similar to a bar chart. The difference is that while the bar chart has spaces between bars, the bars in a histogram are drawn adjacent to each other.
Example 5
We illustrate the construction of a histogram by recalling the frequency distribution in our fisherman example. Below is the frequency distribution.

Table 11.23 Daily fish catch

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>Number of days</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>5</td>
</tr>
<tr>
<td>12 – 19</td>
<td>16</td>
</tr>
<tr>
<td>19 – 26</td>
<td>14</td>
</tr>
<tr>
<td>26 – 33</td>
<td>9</td>
</tr>
<tr>
<td>33 – 40</td>
<td>4</td>
</tr>
<tr>
<td>40 – 47</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
</tr>
</tbody>
</table>

Construct a histogram for the distribution

Solution
Since all the class intervals are then same, therefore the frequencies (number of days) will be represented by the heights of the bars. So we proceed by placing the daily catch values along the x-axis scaled to ensure that it fits the values from 5 to 47 and the frequencies are scaled along the y-axis.

Points to note on the drawing of a histogram
- Each bar represents just one class, the bar width corresponds to the class width i.e. each bar extends from the lower boundary to the upper boundary of the class.
- The bars are joined together (i.e. the values on the x-axis should be continuous)
• If class width vary, then the areas of the bars (width x height) corresponds to the class frequencies. In this case the height of bar, referred to as the frequency density, is given by \( \frac{\text{frequency}}{\text{Width}} = \frac{f}{i} \).

**Examples 6**

An accountant of a progressive tea estate has gathered data on wages paid to tea pickers in the month and has presented the data on a frequency distribution as follows

<table>
<thead>
<tr>
<th>Wage (MK’000)</th>
<th>Number of pickers</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 – 15</td>
<td>20</td>
</tr>
<tr>
<td>15 – 18</td>
<td>12</td>
</tr>
<tr>
<td>18 – 25</td>
<td>35</td>
</tr>
<tr>
<td>25 – 30</td>
<td>27</td>
</tr>
<tr>
<td>30 – 40</td>
<td>8</td>
</tr>
<tr>
<td>40 and over</td>
<td>5</td>
</tr>
</tbody>
</table>

Construct a histogram for the distribution.

**Solution**

Note that the last class is open-ended and that the class widths are not equal. The open ended class is dealt with by assigning it the most common width or the width of the preceding class. In our case we assign it the class width of 5 and the class becomes 40 – 45.

The presence of unequal class width requires that we use frequency densities for the heights of the bars. The frequency densities are calculated in the table below.

<table>
<thead>
<tr>
<th>Wage (MK’000)</th>
<th>f</th>
<th>Frequency density, ( \frac{f}{i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 – 15</td>
<td>20</td>
<td>( \frac{20}{5} = 4 )</td>
</tr>
<tr>
<td>15 – 18</td>
<td>12</td>
<td>( \frac{12}{3} = 4 )</td>
</tr>
<tr>
<td>18 – 25</td>
<td>35</td>
<td>( \frac{35}{7} = 5 )</td>
</tr>
<tr>
<td>25 – 30</td>
<td>27</td>
<td>( \frac{27}{5} = 5.4 )</td>
</tr>
<tr>
<td>30 – 40</td>
<td>8</td>
<td>( \frac{8}{10} = 0.8 )</td>
</tr>
<tr>
<td>40 – 45</td>
<td>5</td>
<td>( \frac{5}{5} = 1 )</td>
</tr>
</tbody>
</table>

Here is the histogram for the distribution.

**Example 7**

Below is age frequency distribution for students from Tayamba Pvt. Secondary School. Represent the distribution on a histogram.
Table 11.26   Students age students

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 – 14</td>
<td>55</td>
</tr>
<tr>
<td>15 – 19</td>
<td>82</td>
</tr>
<tr>
<td>20 – 24</td>
<td>45</td>
</tr>
<tr>
<td>25 – 29</td>
<td>28</td>
</tr>
<tr>
<td>30 – 34</td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td>218</td>
</tr>
</tbody>
</table>

**Solution**

This is case data classes are not continuous. We need therefore to close the gaps between the age groups by converting the class limits into boundaries (see section ____ ) as shown below

Table 11.27   Students age distribution

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Boundaries</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 – 14</td>
<td>9.5 – 14.5</td>
<td>55</td>
</tr>
<tr>
<td>15 – 19</td>
<td>14.5 – 19.5</td>
<td>82</td>
</tr>
<tr>
<td>20 – 24</td>
<td>19.5 – 24.5</td>
<td>45</td>
</tr>
<tr>
<td>25 – 29</td>
<td>24.5 – 29.5</td>
<td>28</td>
</tr>
<tr>
<td>30 – 34</td>
<td>29.5 – 34.5</td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>218</td>
</tr>
</tbody>
</table>
The histogram is presented below.

**Figure 11.10  Students age distribution**

Although a histogram provides a strong visual appeal, one cannot read the exact data values since data values are grouped.

### 11.6.5 Frequency Polygon

A frequency polygon is similar to a histogram. It consists of line segments connecting the points formed by the intersections of the class marks (midpoints) and the class frequencies i.e. connecting the top centres of the bars in a histogram.

To construct a frequency polygon, we scale the class midpoints along the X-axis and the frequencies along the Y-axis. We illustrate the construction of the frequency polygon using the data from the previous example. The data is reproduced below.
Table 11.28  Students age distribution

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 – 14</td>
<td>55</td>
</tr>
<tr>
<td>15 – 19</td>
<td>82</td>
</tr>
<tr>
<td>20 – 24</td>
<td>45</td>
</tr>
<tr>
<td>25 – 29</td>
<td>28</td>
</tr>
<tr>
<td>30 – 34</td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td>218</td>
</tr>
</tbody>
</table>

To construct the frequency polygon we first need to find the class midpoint. The midpoints are given in the following table.

Table 11.29  Students age distribution

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Midpoints</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 – 14</td>
<td>12</td>
<td>55</td>
</tr>
<tr>
<td>15 – 19</td>
<td>17</td>
<td>82</td>
</tr>
<tr>
<td>20 – 24</td>
<td>22</td>
<td>45</td>
</tr>
<tr>
<td>25 – 29</td>
<td>27</td>
<td>28</td>
</tr>
<tr>
<td>30 – 34</td>
<td>32</td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>218</td>
</tr>
</tbody>
</table>

Next we plot the midpoints and their corresponding frequencies on the Cartesian (X-Y) plane. The points to be plotted have the following coordinates (12,55), (17,82), (22,45), (27,28) and (32,8). The points are then connected in order as shown below.

Figure 11.11  Students age distribution-frequency distribution
This does look like a polygon. In order to complete the frequency polygon, the ends must be connected to the x-axis at zero frequencies. To do this, add two imaginary classes, one at either end, and connect the ends of the line graph to the X-axis at their midpoints. In our case the classes are 5 – 9 (midpoint is 7) and 35 – 39 (midpoint is 37). The ends of the line graph will then be connect to the points (7,0) and (37,0) to complete our frequency polygon as shown below.

Figure 11.12

Both the histogram and frequency polygon allow us to get quick picture of the main characteristics of the data i.e. highs, lows, points of concentration etc. However, the frequency polygon has an advantage over the histogram in the sense that it allows for direct comparison of two or more distributions.

Figure 11.13 Age distribution frequency polygons

a)
11.6.6 Ogive
An ogive is a cumulative frequency graph. There are two forms of an ogive and these are: The “less than” ogive and the “more than” ogive. We illustrate the steps followed in constructing an ogive using this example.

Example 8
We shall once again use the data on the daily catches of fish. The distribution is below:

Table 11.30 Daily catch for a fisherman

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>Number of days</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>5</td>
</tr>
<tr>
<td>12 – 19</td>
<td>16</td>
</tr>
<tr>
<td>19 – 26</td>
<td>14</td>
</tr>
<tr>
<td>26 – 33</td>
<td>9</td>
</tr>
<tr>
<td>33 – 40</td>
<td>4</td>
</tr>
<tr>
<td>40 – 47</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
</tr>
</tbody>
</table>
Construct “less-than” and more-than ogives.

**Solution:**
As the description of an ogive suggests, an ogive requires cumulative frequencies. We present the cumulative frequencies in the table below.

<table>
<thead>
<tr>
<th>Daily catch</th>
<th>f</th>
<th>“Less-than” cum frequency, F</th>
<th>“More-than” cum frequency, F</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 12</td>
<td>5</td>
<td>5</td>
<td>50</td>
</tr>
<tr>
<td>12 – 19</td>
<td>16</td>
<td>21</td>
<td>45</td>
</tr>
<tr>
<td>19 – 26</td>
<td>14</td>
<td>35</td>
<td>29</td>
</tr>
<tr>
<td>26 – 33</td>
<td>9</td>
<td>44</td>
<td>15</td>
</tr>
<tr>
<td>33 – 40</td>
<td>4</td>
<td>48</td>
<td>6</td>
</tr>
<tr>
<td>40 – 47</td>
<td>2</td>
<td>50</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The next step is to plot the points. For a less-than ogive, we plot the class upper boundaries and the corresponding cumulative frequencies. In our case the first plot is at (12, 5) and the next is (19, 21). The rest of the points follow in this order (26, 35), (33, 44), (40, 48) and (47, 50). Once plotted the points are connected to produce a less-than ogive. To complete the less-than ogive, the lower suspended end is connected to the X-axis at the lower boundary of the first class. A completed less-than ogive is given below.

Figure 11.14 Daily catch “less than” ogive

For a more-than ogive, we plot the class lower boundaries and the corresponding more-than cumulative frequencies. In our case the first plot would be at (5, 50) and the next is (12, 45). The rest of the points
follow in this order (19,29), (33,6), and (40,2). Once plotted the points are connected to produce a more-than ogive. To complete the more-than ogive, the lower suspended end is connected to the X-axis at the upper boundary of the last class. A completed more-than ogive is given below.

Figure 11.15  Daily catch “more than” ogive

11.7  LORENZ CURVE

The Lorenz curve is a curve that shows how even or uneven the distribution of some variable such as income and wealth is. It is therefore primarily used to give a visual display of the degree of inequality of any given data.

In an ideal situation and given a number of employees at an organisation, we would expect that 20% of the employees would command 20% of the wage bill, 50% would command 50% of the wage bill and so on. The graph of such an ideal situation would look as follows

Figure 11.16  Lorenz curve (diagonal)
However, in reality, incomes and other issues are rarely evenly distributed and the graph would not be a straight line. The graph above represents a line of even distribution or line of equality/uniformity. A Lorenz curve therefore shows how degree of deviation from the line of absolute equality.

**Construct a Lorenz Curve**

We can easily construct a Lorenz curve using the following steps:

1. Calculate cumulative percentage frequency for each class
2. Calculate cumulative percentage class totals for each class
3. Plot the cumulative percentage frequency (Y-axis) against cumulative percentage class totals (X-axis) scaled accordingly.
4. Connect the points with smooth curve to obtain a Lorenz curve
5. Fit in a line equality by joining the origin (0,0) to the point (100,100). This helps to show the degree of inequality

**Example 8**

Laponda Ltd has the following employee and wage distribution. The CEO has been claiming that his organisation does not have huge imbalances in employees pay. Draw a Lorenz curve of the distribution to prove or disprove the CEO’s claim.
Table 11.32 Laponda Wage distribution

<table>
<thead>
<tr>
<th>Salary (K’000)</th>
<th>Number of employees</th>
<th>Total wage bill (MK’000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 – 50</td>
<td>26</td>
<td>1300</td>
</tr>
<tr>
<td>50 – 100</td>
<td>65</td>
<td>9750</td>
</tr>
<tr>
<td>100 – 150</td>
<td>50</td>
<td>12500</td>
</tr>
<tr>
<td>150 – 200</td>
<td>40</td>
<td>14000</td>
</tr>
<tr>
<td>200 – 250</td>
<td>30</td>
<td>13500</td>
</tr>
<tr>
<td>250 – 300</td>
<td>25</td>
<td>13750</td>
</tr>
<tr>
<td>300 – 350</td>
<td>15</td>
<td>9750</td>
</tr>
<tr>
<td>350 – 400</td>
<td>10</td>
<td>7500</td>
</tr>
<tr>
<td>400 – 450</td>
<td>8</td>
<td>6800</td>
</tr>
<tr>
<td>450 – 500</td>
<td>5</td>
<td>4750</td>
</tr>
<tr>
<td>500 – 550</td>
<td>3</td>
<td>3150</td>
</tr>
<tr>
<td>550 – 600</td>
<td>2</td>
<td>2300</td>
</tr>
<tr>
<td>600 – 650</td>
<td>1</td>
<td>1250</td>
</tr>
</tbody>
</table>

Solution
Since the wage bill totals for each class have been given, we do not need to be estimated estimate them. So we proceed to calculate the cumulative percentages.

Table 11.33 Laponda wage and employee distribution

<table>
<thead>
<tr>
<th>Number of employees</th>
<th>Total wage bill (MK’000)</th>
<th>Cum employees, F</th>
<th>F%</th>
<th>Cum total wage bill</th>
<th>Cum total wage bill %</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>1300</td>
<td>26</td>
<td>9.3</td>
<td>1300</td>
<td>1.3</td>
</tr>
<tr>
<td>65</td>
<td>9750</td>
<td>91</td>
<td>32.5</td>
<td>11050</td>
<td>11.0</td>
</tr>
<tr>
<td>50</td>
<td>12500</td>
<td>141</td>
<td>50.4</td>
<td>23500</td>
<td>23.4</td>
</tr>
<tr>
<td>40</td>
<td>14000</td>
<td>181</td>
<td>64.6</td>
<td>37550</td>
<td>37.4</td>
</tr>
<tr>
<td>30</td>
<td>13500</td>
<td>211</td>
<td>75.4</td>
<td>51050</td>
<td>50.9</td>
</tr>
<tr>
<td>25</td>
<td>13750</td>
<td>236</td>
<td>84.3</td>
<td>64800</td>
<td>64.6</td>
</tr>
<tr>
<td>15</td>
<td>9750</td>
<td>251</td>
<td>89.6</td>
<td>74550</td>
<td>74.3</td>
</tr>
<tr>
<td>10</td>
<td>7500</td>
<td>261</td>
<td>93.2</td>
<td>82050</td>
<td>81.8</td>
</tr>
<tr>
<td>8</td>
<td>6800</td>
<td>269</td>
<td>96.1</td>
<td>88850</td>
<td>88.6</td>
</tr>
<tr>
<td>5</td>
<td>4750</td>
<td>274</td>
<td>97.9</td>
<td>93600</td>
<td>93.3</td>
</tr>
<tr>
<td>3</td>
<td>3150</td>
<td>277</td>
<td>98.9</td>
<td>96750</td>
<td>96.5</td>
</tr>
<tr>
<td>2</td>
<td>2300</td>
<td>279</td>
<td>99.6</td>
<td>99050</td>
<td>98.8</td>
</tr>
<tr>
<td>1</td>
<td>1250</td>
<td>280</td>
<td>100.0</td>
<td>100300</td>
<td>100.0</td>
</tr>
</tbody>
</table>
To construct a Lorenz curve, we plot the cumulative frequencies (F%) against the cumulative total wage bill (%) on a Cartesian plane and the join the points. The points to be plotted are: (1.3, 9.3), (11, 32.5), (23.4, 50.4), . . . (98.8, 99.6) and (100, 100). We must remember to include the origin, (0,0) as we plot the points. We will complete the Lorenz curve by fitting in a line of equality or even distribution. The Lorenze curve is given below.

![Lorenz curve](image-url)

**Comments:**

Generally, the closer the Lorenze curve is to the line of even distribution or line of equality, the more even the distribution is. For Laponda Ltd, the Lorenze curve bulges significantly from the line of equality. It is clear that the wage distribution for Laponda is uneven. In fact, we can see from the curve that 30% of the lowly paid employees share just about 10% of the total wage bill.

### 11.8 Z CHART

The Z chart displays time series data showing 3 major aspects. The three aspects when plotted roughly form the letter Z. These aspects are:

i) Actual figures or data values for a chosen period
ii) Cumulative figures or date values for the chosen period
iii) Moving totals to date for the same period

We illustrate the construction of a Z chart using the following example.

**Example 9**

The production manager of a local firm has recorded the following monthly production figures in order to assess the production levels over time.
Table 11.33  Local firm’s monthly production

<table>
<thead>
<tr>
<th></th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>2011</td>
<td>150</td>
<td>154</td>
<td>183</td>
<td>162</td>
<td>181</td>
<td>149</td>
<td>130</td>
<td>152</td>
<td>180</td>
<td>199</td>
<td>193</td>
<td>186</td>
</tr>
<tr>
<td>2012</td>
<td>162</td>
<td>163</td>
<td>171</td>
<td>158</td>
<td>175</td>
<td>145</td>
<td>121</td>
<td>138</td>
<td>172</td>
<td>175</td>
<td>163</td>
<td>152</td>
</tr>
</tbody>
</table>

Construct a Z chart for 2012

**Solution**

The actual production figures for 2012 are shown in Column 2 below. We cumulate the figures for 2012 and show the cumulative figures in Column 3 below. To achieve this, we start with 162 for January, for February we add 163 to 162 and obtain 325, For March 171 to 325 to obtain 496, and so on.

Since there are 12 time points in each period (i.e. year), the moving totals will be calculated using period, $P = 12$.

The moving total for January 2012 is the total from February 2011 to January 2012, i.e. $154 + 183 + 162 + \ldots + 193 + 186 + 162 = 2031$.

The moving total for February 2012 is the total from March 2011 to February 2012, i.e. $183 + 162 + 181 + \ldots + 186 + 162 + 163 = 2040$.

An easier way to find the moving total for February 2012 is $2031 - 154 + 163 = 2040$. If we proceed in this way, we should obtain 1895 for December 2012. This figure must be the same as the cumulative total for December 2012. The moving totals are given in Column 4 below.

Table 11.34  Monthly production, cumulative and moving totals

<table>
<thead>
<tr>
<th>Month</th>
<th>Actual figures 2012</th>
<th>Cumulative figures, 2012</th>
<th>Moving totals 2012</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>162</td>
<td>162</td>
<td>2031</td>
</tr>
<tr>
<td>Feb</td>
<td>163</td>
<td>325</td>
<td>2040</td>
</tr>
<tr>
<td>Mar</td>
<td>171</td>
<td>496</td>
<td>2028</td>
</tr>
<tr>
<td>Apr</td>
<td>158</td>
<td>654</td>
<td>2024</td>
</tr>
<tr>
<td>May</td>
<td>175</td>
<td>829</td>
<td>2018</td>
</tr>
<tr>
<td>Jun</td>
<td>145</td>
<td>974</td>
<td>2014</td>
</tr>
<tr>
<td>Jul</td>
<td>121</td>
<td>1095</td>
<td>2005</td>
</tr>
<tr>
<td>Aug</td>
<td>138</td>
<td>1233</td>
<td>1991</td>
</tr>
<tr>
<td>Sep</td>
<td>172</td>
<td>1405</td>
<td>1983</td>
</tr>
<tr>
<td>Oct</td>
<td>175</td>
<td>1580</td>
<td>1959</td>
</tr>
<tr>
<td>Nov</td>
<td>163</td>
<td>1743</td>
<td>1929</td>
</tr>
<tr>
<td>Dec</td>
<td>152</td>
<td>1895</td>
<td>1895</td>
</tr>
</tbody>
</table>
To construct a Z chart, we shall plot the figures from the above table against time points i.e. months. Each month shall have two data points plotted against it. The Z chart is shown below.

Figure 11.18 Z-chart

Comments on the Z chart:
Actual figures show some fluctuations in the level of production from month to month. The cumulative total shows a steady rise in production. The year to-date (moving totals) figures are indicating that production is declining over time.

CHAPTER SUMMARY

In this chapter we have discussed
- A variety of methods for presenting data pictorially. These include frequency distributions, charts for presenting qualitative data such as the pie chart and bar chart (simple, component, and multiple bar charts), and the pictogram.
- We also discussed graphs that are well suited for quantitative data namely the histogram, the frequency polygon and ogives which are basically cumulative frequency polygons.
- Other curves including the Lorenz curves and Z charts which are primarily used to assess levels of inequality in a distribution and display time series data respectively.
ND OF CHAPTER EXERCISES

The following are sales for XYZ Ltd for the years 2010 to 2012. Present them on a bar chart:

<table>
<thead>
<tr>
<th>Item</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food Item</td>
<td>500</td>
<td>800</td>
<td>1000</td>
</tr>
<tr>
<td>Drinks</td>
<td>600</td>
<td>400</td>
<td>200</td>
</tr>
<tr>
<td>Clothing</td>
<td>400</td>
<td>400</td>
<td>800</td>
</tr>
<tr>
<td>TOTAL</td>
<td>1500</td>
<td>1600</td>
<td>2000</td>
</tr>
</tbody>
</table>

Students at a certain school were surveyed to find out the mode of transport they used when going to school. The results were:

Walking: 9, Bicycle: 10, Car: 6 and Bus: 15.

a) Construct
   i. a pie chart of radius 4cm to present this information
   ii. a simple bar chart of the same data
b) Comment on both the charts

A company decided to research the price of laptops on the market. An analysis of advertisements in the press and specialised IT bulletins produced the following information:

<table>
<thead>
<tr>
<th>Price (MK’000)</th>
<th>Number of laptops</th>
</tr>
</thead>
<tbody>
<tr>
<td>200 – 250</td>
<td>5</td>
</tr>
<tr>
<td>250 – 300</td>
<td>12</td>
</tr>
<tr>
<td>300 – 400</td>
<td>15</td>
</tr>
<tr>
<td>400 – 420</td>
<td>9</td>
</tr>
<tr>
<td>420 – 450</td>
<td>6</td>
</tr>
<tr>
<td>450 – 680</td>
<td>3</td>
</tr>
</tbody>
</table>

raw a histogram of the prices.

The following set of data represents the age distribution of a company’s workforce of 170 employees.

<table>
<thead>
<tr>
<th>Age (Year)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>18 but under 30</td>
<td>56</td>
</tr>
<tr>
<td>30 but under 40</td>
<td>44</td>
</tr>
<tr>
<td>40 but under 50</td>
<td>35</td>
</tr>
<tr>
<td>50 but under 60</td>
<td>27</td>
</tr>
<tr>
<td>60 but under 70</td>
<td>7</td>
</tr>
<tr>
<td>70 and over</td>
<td>1</td>
</tr>
</tbody>
</table>
5. About 60% of small and medium sized businesses are family owned. An international survey asked chief executive officers (CEOs) of family owned businesses how they became CEO. Responses were that the CEO inherited the business; the CEO built the business, or the CEO was hired by the family owned firm. A sample of 26 CEOs of family owned business provided the following data on how each one of them became CEO.

<table>
<thead>
<tr>
<th>Built</th>
<th>Built</th>
<th>Built</th>
<th>Built</th>
<th>Built</th>
<th>Inherited</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inherited</td>
<td>Built</td>
<td>Inherited</td>
<td>Built</td>
<td>Inherited</td>
<td>Inherited</td>
</tr>
<tr>
<td>Inherited</td>
<td>Built</td>
<td>Built</td>
<td>Hired</td>
<td>Built</td>
<td>Built</td>
</tr>
<tr>
<td>Built</td>
<td>Hired</td>
<td>Hired</td>
<td>Built</td>
<td>Inherited</td>
<td>Inherited</td>
</tr>
<tr>
<td>Inherited</td>
<td>Inherited</td>
<td>Inherited</td>
<td>Inherited</td>
<td>Hired</td>
<td></td>
</tr>
</tbody>
</table>

a) Construct a frequency distribution table of the above data
b) Draw a bar chart for the data
c) What could be the main reason a person becomes CEO of a family owned business?

6. A food processor makes chambo fillet from chambo supplied from the lake. The bigger the mass the better (larger and juicer) the fillet and therefore the higher the price. It is known that fish of weight greater than 1 Kg make good fillets which fetch better prices. In order to forecast his revenue he decides to weigh the fish coming in. He asks his buyer to take a sample of 100 fish from different consignments in accordance to a predetermined scientific technique. The results are as follows:

Weights of 100 fish in grams:

| 393 | 1327 | 776 | 1383 | 1179 | 913 | 1958 | 651 | 1307 | 1909 |
| 426 | 592 | 1691 | 412 | 1233 | 1490 | 1010 | 1481 | 1400 | 1280 |
| 722 | 1612 | 628 | 1483 | 718 | 1313 | 642 | 1478 | 630 | 1454 |
| 818 | 881 | 1227 | 929 | 1341 | 865 | 982 | 1488 | 983 | 833 |
| 1145 | 1444 | 634 | 1010 | 1727 | 1076 | 1137 | 1439 | 1149 | 1074 |
| 1296 | 1321 | 1690 | 1364 | 1916 | 1206 | 1309 | 1756 | 1536 | 1366 |
| 1439 | 810 | 1478 | 1532 | 784 | 1182 | 1573 | 429 | 1519 | 1303 |
| 1639 | 1689 | 1083 | 1670 | 1617 | 1610 | 1144 | 1370 | 1762 | 1748 |
| 1840 | 3890 | 806 | 1733 | 1917 | 1435 | 1845 | 1380 | 1631 | 214 |
| 2027 | 1212 | 1332 | 1482 | 1442 | 1553 | 1416 | 1170 | 1763 | 1129 |

a) Suggest suitable classes of a frequency distribution for the weights of fish
b) Construct the frequency distribution with one open ended class and state a reason for your choice
c) Construct a less than ogive of the data
d) Construct a less than cumulative diagram of the weights
7. The following are sales figures of a small scale grocery shop.

<table>
<thead>
<tr>
<th></th>
<th>Actual 2008</th>
<th>Actual 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>150</td>
<td>162</td>
</tr>
<tr>
<td>February</td>
<td>154</td>
<td>163</td>
</tr>
<tr>
<td>March</td>
<td>183</td>
<td>171</td>
</tr>
<tr>
<td>April</td>
<td>162</td>
<td>158</td>
</tr>
<tr>
<td>May</td>
<td>181</td>
<td>175</td>
</tr>
<tr>
<td>June</td>
<td>149</td>
<td>145</td>
</tr>
<tr>
<td>July</td>
<td>130</td>
<td>121</td>
</tr>
<tr>
<td>August</td>
<td>152</td>
<td>138</td>
</tr>
<tr>
<td>September</td>
<td>186</td>
<td>172</td>
</tr>
<tr>
<td>October</td>
<td>199</td>
<td>175</td>
</tr>
<tr>
<td>November</td>
<td>193</td>
<td>163</td>
</tr>
<tr>
<td>December</td>
<td>168</td>
<td>152</td>
</tr>
</tbody>
</table>

a) Present the figures on a Z chart
b) Comment on what the chart shows

8. The following is a distribution of monthly incomes among farmers in Maganga Village.

<table>
<thead>
<tr>
<th>Incomes (MK'000)</th>
<th>Number of farmers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Women</td>
</tr>
<tr>
<td>Less than 30</td>
<td>2</td>
</tr>
<tr>
<td>30 up to 40</td>
<td>3</td>
</tr>
<tr>
<td>40 up to 50</td>
<td>17</td>
</tr>
<tr>
<td>50 up to 60</td>
<td>17</td>
</tr>
<tr>
<td>60 up to 70</td>
<td>8</td>
</tr>
<tr>
<td>70 up to 80</td>
<td>3</td>
</tr>
<tr>
<td>80 or more</td>
<td>0</td>
</tr>
</tbody>
</table>

Construct separate Lorenz curves for women and men on the same Cartesian plane. Comment on the distribution of income among women and men in Maganga.

9. The following data shows the distribution of wealth in Chindozwa.

<table>
<thead>
<tr>
<th>Wealth owned by:</th>
<th>Percentage of total wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>most wealth 1%</td>
<td>18</td>
</tr>
<tr>
<td>most wealth 5%</td>
<td>37</td>
</tr>
<tr>
<td>most wealth 10%</td>
<td>51</td>
</tr>
<tr>
<td>most wealth 25%</td>
<td>72</td>
</tr>
<tr>
<td>most wealth 50%</td>
<td>93</td>
</tr>
<tr>
<td>all wealth owners</td>
<td>100</td>
</tr>
</tbody>
</table>

Represent the information on a Lorenz curve. Briefly comment on your diagram.
CHAPTER 12  STATISTICAL MEASURES

LEARNING OBJECTIVES

By the end of this chapter students should be able to:

i. Calculate measures of central tendency from simple data
ii. Calculate measures of central tendency for frequency distributions
iii. Interpret measures of central tendency
iv. Calculate measures of dispersion
v. Interpret measures of dispersions
vi. Compare distributions using summary measures
vii. Determine the skewness of a distribution
viii. Interpret the skewness of a distribution

12.0  INTRODUCTION

Data summarization is a process of combining scores into a single number, called a statistic. Statistics serve two functions: they estimate parameters in population models and they describe the data. This chapter looks at measures that typify the data, called the measures of central tendency or averages, and measures of dispersion which show how compact or spread the data values are.

12.1  MEASURES OF CENTRAL TENDENCY

The commonly used measures of central tendency include:

- Arithmetic mean
- The mode
- The median
- Geometric Mean

12.1.1  The Arithmetic Mean

The Arithmetic mean, commonly called mean, is the sum of all scores in a data set (distribution) divided by the total number of such scores.

\[ \text{Mean} = \frac{x_1 + x_2 + \ldots + x_n}{n} \]

Notation:
The arithmetic mean is usually denoted by \( \bar{x} \) and the sum

\[ x_1 + x_2 + \ldots + x_n = \sum x \]

Hence mean:

\[ \bar{x} = \frac{\sum x}{n} \]
Example 1
Given data values (x) to be 4, 2, -3, 7, 5 find the mean.

Solution
Mean: \[ \bar{x} = \frac{\sum x}{n} \]
\[ = \frac{4 + 2 + (-3) + 7 + 5}{5} \]
\[ = \frac{15}{5} \]
\[ = 3 \]

Mean for frequency distributions
When data is in form of a frequency distribution, the mean is calculated using the following formula.

\[ \bar{x} = \frac{\sum fx}{\sum f} \]

Where \( x \) are the data values and \( f \) their corresponding frequencies.

Note that for grouped frequencies distributions the data values (x) are the class midpoints.

Example 2
The values 0, 1, 2, 3, 4, and 5 have been presented in a simple frequency distribution as follows:

Table 12.1

<table>
<thead>
<tr>
<th>Data values(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Calculate the arithmetic mean

Solution

Table 12.2 Table of calculations

<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
<th>fx</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>24</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

\[ \bar{x} = \frac{\sum fx}{\sum f} \]

\[ 85 \]
Note that the above is a simple frequency distribution where values of x can easily be identified.

In the following example, values of x are not identifiable. The calculation of mean in the following will need the estimation of x because x is lost in grouping the data.

**Example 3**
Calculate the arithmetic mean of the following

<table>
<thead>
<tr>
<th>Order</th>
<th>Frequency</th>
<th>x</th>
<th>fx</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-10</td>
<td>1</td>
<td>7.5</td>
<td>7.5</td>
</tr>
<tr>
<td>10 - 15</td>
<td>5</td>
<td>12.5</td>
<td>62.5</td>
</tr>
<tr>
<td>15 - 20</td>
<td>7</td>
<td>17.5</td>
<td>122.5</td>
</tr>
<tr>
<td>20 - 25</td>
<td>10</td>
<td>22.5</td>
<td>225</td>
</tr>
<tr>
<td>25 - 30</td>
<td>7</td>
<td>27.5</td>
<td>192.5</td>
</tr>
<tr>
<td>30 - 35</td>
<td>4</td>
<td>32.5</td>
<td>130</td>
</tr>
<tr>
<td>35 - 40</td>
<td>3</td>
<td>37.5</td>
<td>112.5</td>
</tr>
<tr>
<td>40 - 45</td>
<td>2</td>
<td>42.5</td>
<td>85</td>
</tr>
<tr>
<td>45 - 50</td>
<td>1</td>
<td>47.5</td>
<td>47.5</td>
</tr>
</tbody>
</table>

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>40</td>
<td>985</td>
<td></td>
</tr>
</tbody>
</table>

x values are estimated by the midpoints of each class of data

\[
\text{Mean } = \bar{x} = \frac{\sum fx}{\sum f} = \frac{985}{40} = 24.625
\]
Example 4:

Mr. Selemani has 60 of salesmen in his business empire. He is interested in finding out the average sales they make and to do so he has tallied the number of sales each one makes and has grouped them as follows:

Table 12.5 Sales

<table>
<thead>
<tr>
<th>Number of sales</th>
<th>Number of salesmen</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-4</td>
<td>1</td>
</tr>
<tr>
<td>5-9</td>
<td>10</td>
</tr>
<tr>
<td>10-14</td>
<td>18</td>
</tr>
<tr>
<td>15-19</td>
<td>16</td>
</tr>
<tr>
<td>20-24S</td>
<td>11</td>
</tr>
<tr>
<td>25 and above</td>
<td>4</td>
</tr>
</tbody>
</table>

Find the arithmetic mean of the sales for the salesmen as a suitable average.

Solution

Note that the frequency distribution has an open ended class (25 and above). This is closed by assigning it the most common class interval. In this case 4 and it becomes 25 to 29.

Table 12.6 Sales (table of calculations)

<table>
<thead>
<tr>
<th>Number of sales</th>
<th>x</th>
<th>f</th>
<th>fx</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 4</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5 - 9</td>
<td>7</td>
<td>10</td>
<td>70</td>
</tr>
<tr>
<td>10 - 14</td>
<td>12</td>
<td>18</td>
<td>216</td>
</tr>
<tr>
<td>15 - 19</td>
<td>17</td>
<td>16</td>
<td>272</td>
</tr>
<tr>
<td>20 - 24</td>
<td>22</td>
<td>11</td>
<td>242</td>
</tr>
<tr>
<td>25 - 29</td>
<td>27</td>
<td>4</td>
<td>108</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td></td>
<td>910</td>
</tr>
</tbody>
</table>

\[
\bar{x} = \frac{\sum fx}{\sum f}
\]

\[
= \frac{910}{60}
\]

\[
= 15.16667
\]

The average number of sales per salesman is 15

Advantages and disadvantages of the Arithmetic Mean

The arithmetic mean has the following advantages and disadvantages.
Advantages

(i) The mean is easy to understand
(ii) It is a true representative of the data (it uses all data values)
(iii) The mean can be used in further statistical analysis.

Disadvantages

(i) The mean is influenced by the extreme values
(ii) It can result in unrealistic figures

Note: These advantages and disadvantages help the user to decide whether or not the mean is the appropriate average.

12.1.2 The Mode

Definition

Mode of a distribution of data is the data item that has the highest frequency of occurrence

A. Mode for ungrouped data or a simple frequency distribution.
If data is not grouped, the mode can be found by inspection or simple count of the individual values. The same applies when data is in form of a simple frequency distribution.

Example 5
Given x values as: 4, 0, 2, 5, 4, 8, 3. Find the mode

Solution: By inspection mode = 4

Example 6
Given x =4, 0, 2, 5, 4, 2, 8, 3

Solution: Mode = 4 and 2

Note that it is possible for a distribution to have more than one mode as in the second example. A distribution with one mode is said to be unimodal and bimodal when it has two modes.

Example 7
Consider the simple frequency distribution given below and find the mode.

Table 12.7 table of x values

<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Solution The mode is 3 (item with the highest frequency)
B. Mode for grouped data (Using a formula)

When data is grouped, the mode can be estimated by using a formula or histogram.

The starting point is to identify the modal class. Given that individual data items of a frequency distribution cannot be identified, the mode is assumed to be contained in the class with the highest frequency. Having identified the modal class, the mode can then be estimated using the formula:

$$Mode = L + \left(\frac{f_1 - f_0}{2f_1 - f_2 - f_0}\right) C$$

Where

- $f_1$ = frequency of modal class
- $f_0$ = the frequency immediately before the modal class
- $f_2$ = frequency immediately after the modal class
- $C$ = modal class width
- $L$ = lower boundary of modal class

Example 8

Kamkaka Farms supplies milk directly to shops, hotels and lodges in the Capital City. The MD is interested in the most likely number of orders that the farm receives in order to prepare for the supplies realistically.

The following are the figures tabulated by the sales office. Calculate the mode as his guide.

Table 12.8 Weekly orders

<table>
<thead>
<tr>
<th>Orders in a week</th>
<th>Number of weeks</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-15</td>
<td>6</td>
</tr>
<tr>
<td>15-20</td>
<td>7</td>
</tr>
<tr>
<td>20-25</td>
<td>8</td>
</tr>
<tr>
<td>25-30</td>
<td>10</td>
</tr>
<tr>
<td>30-35</td>
<td>5</td>
</tr>
<tr>
<td>35-40</td>
<td>3</td>
</tr>
</tbody>
</table>

Solution

First identify the modal class

Highest frequency = 10.

Modal = 25 - 30.

$L$ (Lower limit of modal class) = 25

$f_1$ (the frequency of the modal class) = 10

$f_0$ (The frequency of class immediately before modal class) = 8

$f_2$ (Frequency of the class immediately after the modal class) = 5

$C$ = modal class width = 30 – 25 = 5

$$Mode = L + \left(\frac{f_1 - f_0}{2f_1 - f_2 - f_0}\right) C$$

$$= 25 + \left(\frac{10 - 8}{2 	imes 10 - 5 - 8}\right) \times 5$$

$$= 25 + 0.625 \times 5$$

$$= 26.43$$
C. Mode for grouped data (Using histogram)

Example 9

Consider Kamkaka’s figures above. Draw the histogram to represent the number of orders and use it to estimate the mode.

Solution:

Figure 12.1  Mode using histogram

Note the two lines drawn in the modal class bar. The mode is estimated by dropping a vertical line from the intersection point of the two lines to the x-axis

Advantages and Disadvantages of the Mode

Advantages of the mode
i) It is easy to understand
ii) The mode is not easily influenced by extreme values
iii) The mode is realistic (it gives realistic figures) and it will usually be one of the data values except for the ones estimated through formula or histogram.

Disadvantages
i) The mode cannot be used in further statistical analysis it does not use all the data values.
ii) The mode may not be unique.
12.1.3 Median

**Definition**

If data is arranged in some ascending or descending order the median is the data item at the middle of the distribution. Alternatively the median can be described as the numerical value separating the higher half of a data in a distribution from the lower half.

**Example 10**

Given the x-values as: 3, 10, 4, 5, 8, 15, 3. Find the median

**Solution:**
The data rearranged: 3 3, 4, 5, 8, 10, 15

The value in the middle is **5**. This is the median

**Methods of finding the median**

In example 10 the median was determined by a simple count because the number of data items n is small. If, however, n is large the following techniques can be used to determine the median:

a. If n is odd the median is the item on position \( \frac{n+1}{2} \) when the values data values are arranged in order.

**Example 11**

Use the technique outlined above to determine the median of example 1.

**Solution:** \( n = 7 \)

The median is therefore the item on position \( \frac{n+1}{2} = 4 \)

Rearranged data: 3, 3, 4, 5, 8, 10, 15

The 4th item is 5

Hence median = 5

b. If n is even the median is the arithmetic mean of the items on positions \( \frac{n}{2} \) and \( \frac{n}{2} + 1 \)

**Example 12**

Given the x-values as: 22, 10, 4, 5, 8, 15, 3, 17. Find the median

**Solution**

\( n = 8 \), which is even

Therefore we find items on positions \( \frac{8}{2} = 4 \) and \( \frac{8}{2} + 1 \)

Rearranged data: 3, 4, 5, 8, 10, 15, 17, 22

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The 4th and 5th items are 8 and 10
Median \(= \frac{8+10}{2} = 9\)

Example 13

Consider the following simple frequency distribution

Table 12.9 | x values
---|---
<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

Find the median.

Solution

The distribution shows that the x values are already arranged in an ascending order. A cumulative frequency column can be used to count the values and determine the median.

Table 12.10 | Table of calculations |
---|---|
<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
<th>Cum f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>22</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>29</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>33</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>34</td>
</tr>
</tbody>
</table>

Since in a grouped distributions individual values are lost, the median can only be estimated and they are two ways this which this can be done

(a) Using formula

Given a frequency distribution the median is estimated by the interpolation formula:

\[
\text{Median} = L + \left( \frac{\Sigma f_i}{2} - \frac{\Sigma f}{2} \right) \frac{C}{f}
\]

Having already identified a median class,
L= Lower boundary of the median class
\[ f = \text{Frequency of the median class} \]
\[ C = \text{median class width} \]
\[ \Sigma f_0 = \text{Cumulative frequency up to the class immediately before the median class} \]

**Example 14** Consider the following table regarding Kamkaka’s orders

<table>
<thead>
<tr>
<th>Orders in a week</th>
<th>Number of weeks</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-15</td>
<td>6</td>
</tr>
<tr>
<td>15-20</td>
<td>7</td>
</tr>
<tr>
<td>20-25</td>
<td>8</td>
</tr>
<tr>
<td>25-30</td>
<td>10</td>
</tr>
<tr>
<td>30-35</td>
<td>5</td>
</tr>
<tr>
<td>35-40</td>
<td>3</td>
</tr>
</tbody>
</table>

Estimate the median number of orders

**Solution:**

<table>
<thead>
<tr>
<th>Orders in a week</th>
<th>Number of weeks</th>
<th>Cum f</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>15</td>
<td>20</td>
<td>7</td>
</tr>
<tr>
<td>20</td>
<td>25</td>
<td>8</td>
</tr>
<tr>
<td>25</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>30</td>
<td>35</td>
<td>5</td>
</tr>
<tr>
<td>35</td>
<td>40</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ \frac{\Sigma f}{2} = 19.5 \]

Cumulative frequency 19.5 falls in class 20 to 25. This is the median class

\[ \text{Median} = L + \left( \frac{\frac{\Sigma f}{2} - \Sigma f_0}{f} \right) C \]

\[ L = 20 \quad f = 8 \quad C = 5 \]
\[ \Sigma f_0 = 13 \]

\[ \text{Median} = 20 + \left( \frac{19.5 - 13}{8} \right) 5 \]

\[ = 20 + \left( \frac{6.5}{8} \right) \times 5 \]

\[ = 20 + 4.06 \]

\[ = 24.06 \]

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(b) Using a cumulative frequency curve (Ogive).

The median can also be estimated graphically by using the ogive.

Example 15

Consider the orders received by Kamkaka farms in example 4. Draw an ogive for the distribution of the orders and use it to estimate the median.

Solution

<table>
<thead>
<tr>
<th>Orders</th>
<th>f</th>
<th>Cum f</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>&quot;Less than&quot;</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>15</td>
<td>20</td>
<td>7</td>
</tr>
<tr>
<td>20</td>
<td>25</td>
<td>8</td>
</tr>
<tr>
<td>25</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>30</td>
<td>35</td>
<td>5</td>
</tr>
<tr>
<td>35</td>
<td>40</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 12.2 The ogive and median

Advantages and disadvantages of the median

Advantages
It is easy to understand

ii. It is not influenced by extreme values

iii. It will often assume a value equal to the original data

Disadvantages

i. It cannot be used in advanced statistical analysis

ii. It does not use all the data values and therefore it’s not truly representative of the distribution.

iii. For ungrouped data, the median requires that data to be rearranged.

12.1.4 Geometric Mean

Unlike the arithmetic mean which uses sums of data values, the geometric mean uses products of the data values. For a data set with \( n \) values, the geometric mean is the \( n^{th} \) root of the product of the data values.

i.e. If \( x_1, x_2, x_3, \ldots, x_n \) is a set of \( n \) data values, then their geometric mean is given by

\[
GM = \sqrt[n]{x_1 \times x_2 \times x_3 \times \ldots \times x_n}
\]

Notation

If \( x_1, x_2, x_3, \ldots, x_n \) is a general set of \( n \) values, then the product \( x_1 \times x_2 \times x_3 \times \ldots \times x_n \) is denoted \( \prod_{i=1}^{n} x_i \)

hence the geometric mean

\[
GM = \left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}
\]

Example 16

Find the geometric mean of 4, 5, 12, 8, and 2

Solution

\[
GM = \sqrt[5]{4 \times 5 \times 12 \times 8 \times 2}
\]

Since there are 5 values, the geometric mean is the 5\(^{th} \) root of their product.

\[
GM = \sqrt[5]{3840} = 5.21034
\]

12.2 MEASURES OF DISPERSION

The measures of central tendency we have looked at in the previous chapter are used to represent data or compare them, but on their own they do not give sufficient information about distributions. Here we will consider the following measures of dispersion.

i. Range

ii. Mean deviation

iii. Variance and standard deviation
v. Coefficient of skewness

12.2.1 Range

Definition

The range of a distribution of data is defined as the difference between the highest value and lowest value in a data set

\[ \text{i.e. Range} = \text{The highest value – the lowest value} \]

Example 17  Find range, given the following values of \( x \)

a)  5, 4, 15, 7, 8, 2, 1
b)  5, 4, -50, 7, 8, 2, 1

Solution  a) Highest value = 15;  Lowest value =1
Range = 15 – 1=14
b) Range =  8  -  (-50)  =  58

12.2.2 Mean Deviation

Definition

The mean deviation of a data items is defined as the arithmetic mean of the absolute differences between the arithmetic mean and each of the data value.

Thus, if \( \bar{x} \) is the mean of some \( x \) values then

\[
\text{Mean Deviation} = \frac{\sum |x - \bar{x}|}{n}
\]

Where \( n \) is the number of data values.

Example 18

The following are ages (in years) of pupils in standard three:11, 7, 6, 8, 10, 7, 9, 8, 6, and 8. Calculate the mean deviation of the ages

Solution

\[
\bar{x} = \frac{11 + 7 + 6 + 8 + 10 + 7 + 9 + 8 + 6 + 8}{10} = \frac{80}{10} = 8
\]

\[
\text{Mean Deviation} = \frac{10}{10} = \frac{3 + 1 + 2 + 0 + 2 + 1 + 1 + 0 + 2 + 0}{10}
\]

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The mean deviation is simple to understand but the problem is that one has to remember to use only the absolute values (ignoring negative signs).

12.2.3 Variance and Standard Deviation

Definition
Standard deviation is a statistic that measures the spread of data around the arithmetic mean. The variance is the square of the standard deviation. It is normal as it will be seen below to calculate the variance first then the standard deviation. The variance is generally taken to be an interim measure step in the calculation of standard deviation.

The standard deviation which is often denoted by $S$ is calculated as:

$$S = \sqrt{\frac{\sum(x - \bar{x})^2}{n}}$$ for simple data.

For data in a frequency distribution

$$S = \sqrt{\frac{\sum f x^2 - (\sum f x)^2}{\sum f}}$$

Example 19

Find the standard deviation of 4, 3, 5, 7, 6

Solution

$$S = \sqrt{\frac{\sum(x - \bar{x})^2}{n}}$$

$$\bar{x} = \frac{4 + 3 + 5 + 7 + 6}{5} = 5$$

$$S = \sqrt{\frac{(4 - 5)^2 + (3 - 5)^2 + (5 - 5)^2 + (7 - 5)^2 + (6 - 5)^2}{5}} = 2$$

Note: For easy computation Standard deviation for simple data is also calculated using the formula:

$$\sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$
Example 20
A day care facility in a small town takes care of 30 under-five children of working class parents. The ages of the children are distributed as follows

Table 12.13  Children’s Age distribution

<table>
<thead>
<tr>
<th>Age of children (x)</th>
<th>Number of children</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>30</strong></td>
</tr>
</tbody>
</table>

Find the standard deviation of the children’s ages.

Solution:

Table 12.14  Table of calculations

<table>
<thead>
<tr>
<th>Age of children (x)</th>
<th>Number of children (f)</th>
<th>fx</th>
<th>fx²</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>30</td>
<td>90</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>20</td>
<td>80</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>30</strong></td>
<td><strong>81</strong></td>
<td><strong>257</strong></td>
</tr>
</tbody>
</table>

\[
\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}
\]

\[
= \sqrt{\frac{257}{30} - \left(\frac{81}{30}\right)^2}
\]

\[
= 1.13 \text{ years}
\]

Example 21
The following set of data represents a frequency distribution of accidents recorded on 50 road stretches selected throughout the country in the month of December.

Table 12:15  Accidents recorded

<table>
<thead>
<tr>
<th>No. of accident recorded</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 but under 2</td>
<td>5</td>
</tr>
<tr>
<td>2 but under 4</td>
<td>10</td>
</tr>
<tr>
<td>4 but under 6</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>166</td>
</tr>
</tbody>
</table>
6 but under 8  
8 but under 10  
10 but under 12

(a) Calculate the mean number of accidents

(b) Calculate the standard deviation and interpret it.

Solution:

Table 12.16 Table of calculations (accidents)

<table>
<thead>
<tr>
<th>Recorded accidents</th>
<th>x</th>
<th>f</th>
<th>( \sum fx )</th>
<th>( \sum fx^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>15</td>
<td>45</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>10</td>
<td>50</td>
<td>250</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>10</td>
<td>70</td>
<td>490</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>6</td>
<td>54</td>
<td>486</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>4</td>
<td>44</td>
<td>484</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>35</strong></td>
<td><strong>233</strong></td>
<td><strong>1755</strong></td>
<td></td>
</tr>
</tbody>
</table>

\( \bar{x} = \frac{\sum fx}{\sum f} \)

\( \bar{x} = \frac{233}{35} = 6.66 \)

\( S = \sqrt{\frac{\sum fx^2}{\sum f} - \left( \frac{\sum fx}{\sum f} \right)^2} \)

\( S = \sqrt{\frac{1755}{35} - \left( \frac{233}{35} \right)^2} = 2.406 \)

The mean number of accidents is 6.66. However the number of accidents spread about the mean by 2.41 accidents.

### 12.2.4 Quartile Deviation

**Definition**

The quartile deviation (sometimes called the semi inter-quartile range) is a value that measures the spread of data around the median. To calculate the quartile deviation one must first of all estimate the quartiles.

Quartiles are data values that divide a distribution into sections of data of 25% (quarter) of the data each.
Illustration

Figure 12.3 Quartiles

<table>
<thead>
<tr>
<th>25%</th>
<th>25%</th>
<th>25%</th>
<th>25% of data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>Q2</td>
<td>Q3</td>
<td></td>
</tr>
</tbody>
</table>

Note that Q2 is the median.

Example 22
Identify the three quartile points in the following set of data values

\[ X: \quad 2, \quad -4, \quad 10, \quad 20, \quad 12, \quad 8, \quad 5, \quad 25, \quad 17, \quad 11, \quad 3, \quad 19, \quad 26 \]

Solution
Rearranged: \(-4, \quad 2, \quad 5, \quad 10, \quad 11, \quad 12, \quad 17, \quad 19, \quad 20, \quad 25, \quad 26\)

\[ Q1 = 5, \quad Q2 = 12, \quad Q3 = 20 \]

Calculation of the quartile deviation

The quartile deviation of data is calculated as; half the difference between the 3rd and 1st quartiles.

If we can denote the quartile deviation with \( QD \)

Then

\[ QD = \frac{Q_3 - Q_1}{2} \]

where \( Q_1 = \text{1st quartile} \)
\( Q_2 = \text{3rd quartile} \)

Example 23
Consider the data in example 6 above and calculate the quartile deviation

Solution:

\[ QD = \frac{Q_3 - Q_1}{2} \]

\[ = \frac{20 - 5}{2} \]
\[ = 7.5 \]

Quartile deviation for grouped data

Example 24
Consider the following delivery times (days) of orders.
Table 12.17  Deliver time distribution

<table>
<thead>
<tr>
<th>Delivery time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>No of orders</td>
<td>4</td>
<td>8</td>
<td>11</td>
<td>12</td>
<td>21</td>
<td>15</td>
<td>10</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Find the (a) the median and (b) the quartile deviation for the delivery times

Solution

Table 12.18  Delivery times-Cumulative frequency distribution

<table>
<thead>
<tr>
<th>X</th>
<th>f</th>
<th>Cum. f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>23</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>35</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td>56</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>71</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>81</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>85</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>87</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>89</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>90</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>91</td>
</tr>
</tbody>
</table>

b. Median:

n is odd and therefore median is item on position

\[ \frac{\sum f + 1}{2} = \frac{91+1}{2} = 46^{th} \text{ item.} \]

From cumulative frequencies, 46\textsuperscript{th} item is in the group that takes the frequencies from a cumulative number of 35 to 56.

This data item is 4.

Median \[ = 4 \]

c. The quartile deviation:

There is need to find the \( Q_1 \) and \( Q_3 \) items

\( Q_1 \) the item on position \[ \frac{n+1}{4} = \frac{91+1}{4} \]

\[ = 23^{\text{rd}} \text{ item} \]
$Q_1 = 2$ (using cum frequencies as a guide)

$Q_3$ is item on position $\frac{3(91+1)}{4} = 69^{th}$ item

$Q_3 = 5$

Therefore $QD = \frac{5-2}{2} = 1.5$

**Example 25**

Consider the orders received by Kamkaka farms in example 4 of section 5 above.

a) Calculate the quartile deviation using a formula.

b) Estimate the quartile deviation graphically

**Solution:**

### Table 12.19 Orders less than cumulative frequencies

<table>
<thead>
<tr>
<th>Orders less than</th>
<th>f</th>
<th>Cum f</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>&quot;Less than&quot;</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>Q1 class</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>Q2 class</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>Q3 class</td>
<td>25</td>
<td>30</td>
</tr>
<tr>
<td>Q4 class</td>
<td>30</td>
<td>35</td>
</tr>
<tr>
<td>35</td>
<td>40</td>
<td>39</td>
</tr>
</tbody>
</table>

$Q_1 = L + \left( \frac{\sum f / 4 - \Sigma f_0}{f} \right) C$

Where $L$ = lower limit of Q1 class  
$f$ = frequency of Q1 class  
$\Sigma f_0$ = Cum f up to class before Q1 class.  
$Q_1$ class is on position $\frac{\sum f}{4} = \frac{39}{4} = 9.75$

$Q_1 = 15 + \left( \frac{9.75-6}{7} \right) 5$

$= 17.68$

$Q_3 = L + \left( \frac{3 \sum f / 4 - \Sigma f_0}{f} \right) C$

Where $L$ = lower limit of class $Q_3$  
$f$ = frequency of $Q_3$ class  
$\sum f_0$ = Cum f up to $Q_2$ class before class
\[ Q_3 \text{ class is on position } \frac{3\sum f}{4} = \frac{3\times39}{4} = 29.25 \]

\[ Q_3 = 25 + \left( \frac{3\times9.75-21}{10} \right) 5 = 29.13 \]

\[ Q_D = \frac{Q_3 - Q_1}{2} = \frac{29.13 - 17.68}{2} = 5.72 \]

- Using the graph requires first drawing the Ogive and locating the \( Q_1 \) and \( Q_3 \) values.

\( Q_3 \) is positioned on \( \frac{3\times39}{4} = 9.75 \).

Draw a horizontal line from 9.75 to the graph. Then drop a perpendicular line from where line above meets the graph.

\( Q_1 \) is where the perpendicular line meets the x axis.

c) From the graph, median = 23.5

d) For the quartile deviation the process is the same as locating the median. The difference is that for the quartile deviation indicators are \( \frac{1}{4} \) of the frequencies (to find \( Q_1 \) and \( \frac{3}{4} \) of the cumulative frequencies to find \( Q_3 \).

\[ Q_1 \text{ is positioned on } \frac{\sum f}{4} = \frac{39}{4} = 9.75 \]

Draw a horizontal line from 9.75 to the graph. Then drop a perpendicular line from where line above meets the graph.

\( Q_1 \) is where the perpendicular line meets the x axis.
And this point is 16.5

\( Q_3 \) is positioned on \( \frac{3 \times \sum f}{4} = \frac{3 \times 39}{4} = 29.25 \)

The process is the same as for \( Q_1 \)

Therefore \( Q_3 = 26.5 \)

\[
QD = \frac{26.5 - 16.5}{2} = 5
\]

12.2.5 Coefficient

Definition

A Coefficient of variation (CV) of a distribution is a measure of the relative spread of data in distributions. It basically expresses the standard deviation as a proportion of the mean. That way it can be used as tool for comparing the spread of two or more distributions.

For a data set with mean \( \bar{x} \) and standard deviation \( s \) the Coefficient of Variation is given by:

\[
CV = \frac{s}{\bar{x}}
\]

Example 26

An auditor examines two batches of invoices. To get a feel of the values he tabulates the values and calculates the mean and standard deviations as follows:

<table>
<thead>
<tr>
<th></th>
<th>Distribution 1</th>
<th>Distribution 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1260</td>
<td>25.2</td>
</tr>
<tr>
<td>Std Deviation</td>
<td>52</td>
<td>15.2</td>
</tr>
</tbody>
</table>

Calculate the coefficients of variation and interpret.

Solution

\[
CV_1 = \frac{s}{\bar{x}} = \frac{52}{1260} = 0.041
\]

\[
CV_2 = \frac{s}{\bar{x}} = \frac{25.2}{15.2} = 0.603
\]

Distribution 2 has a higher CV, therefore it is more variable.
Example 27

A production manager orders an item to be used in production from two suppliers, as a way of ensuring consistent supply. In order to judge the reliability in terms of delivery times, he has compiled the time in days it took to receive 10 orders in the past two weeks from each supplier. The figures are as follows:

Table 12.20 Supplier delivery times

<table>
<thead>
<tr>
<th>Order No.</th>
<th>Supplier A delivery time</th>
<th>Supplier B delivery time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>18</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>19</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>

Advise the manager on the reliability of the suppliers.

Solution:

The reliability here can be judged from mean, standard deviation and the coefficient of variation.

Table 12.21 Table of calculations (delivery times)

<table>
<thead>
<tr>
<th>X</th>
<th>X^2</th>
<th>X</th>
<th>X^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>169</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>16</td>
<td>256</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>25</td>
<td>625</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>81</td>
<td>7</td>
<td>49</td>
</tr>
<tr>
<td>15</td>
<td>225</td>
<td>15</td>
<td>225</td>
</tr>
<tr>
<td>19</td>
<td>361</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>49</td>
<td>18</td>
<td>324</td>
</tr>
<tr>
<td>18</td>
<td>324</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>19</td>
<td>361</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>9</td>
<td>81</td>
</tr>
<tr>
<td>Totals</td>
<td>151</td>
<td>75</td>
<td>837</td>
</tr>
</tbody>
</table>

Supplier A

Mean:
\[ \bar{x} = \frac{151}{10} = 15.1 \]
Standard deviation: 

\[ S = \sqrt{\frac{2551}{10} - 15.1^2} = \sqrt{27.09} = 5.20 \]

Coefficient of variation: 

\[ CV = \frac{5.20}{15.1} = 0.34 \]

Supplier B  
Mean:  
\[ \bar{x} = \frac{75}{10} = 7.5 \]

Standard deviation:  
\[ S = \sqrt{\frac{337}{10} - 7.5^2} = \sqrt{27.45} = 5.24 \]

Coefficient of variation:  
\[ CV = \frac{5.24}{7.5} = 0.70 \]

Supplier B has a lower average meaning the average delivery time is lower, in other words he can deliver quicker on average.

The standard deviations are more or less the same (5.2), However the CV for B is almost two times that of A. This means the delivery times for B are relatively more variable. This makes them less predictable therefore, not very reliable.

12.2.6 Skewness  
A distribution in which the values of mean, median and mode coincide (i.e. mean = median = mode) is known as a symmetrical distribution. Conversely, when values of mean, median and mode are not equal the distribution is known as asymmetrical or skewed distribution. In moderately skewed or asymmetrical distribution a very important relationship exists among these three measures of central tendency. In such distributions the distance between the mean and median is about one-third of the distance between the mean and mode.

\[ \text{Mode} = \text{mean} - 3 [\text{mean} - \text{median}] \]

\[ Mode = 3 \text{Median} - 2 \text{Mean} \]

\[ \text{Median} = \text{mode} + (2/3)(\text{mean} - \text{mode}) \]

Example 28  
Given median = 20.6, mode = 26. Find mean.

Solution:  
Mode = 3 Median – 2 Mean
Pearson’s measure of skewness

While the degree of skewness could be measured by the difference between the mean and mode, for most practical purposes it is required that a measure of skewness be unit free. The pearson’s measure of skewness is used in this regard.

Pearson’s measure of skewness:

\[ P_{sk} = \frac{\text{Mean} - \text{Mode}}{\text{Standard deviation}} \]

\[ = \frac{3(\text{Mean} - \text{Mode})}{\text{Standard deviation}} \]

If \( P_{sk} > 0 \) the distribution is said to be positively skewed

\( P_{sk} < 0 \) the distribution is said to be negatively skewed

\( P_{sk} = 0 \) there is no skewness (the distribution is symmetric)

Example 29

Given that the distribution in the above example (with median = 20.6, mode = 26) has a standard deviation of 6.1 calculate the Pearson’s Measure of skewness

Solution.

\[ P_{sk} = \frac{\text{Mean} - \text{Mode}}{\text{standard deviation}} \]

\[ = \frac{17.9 - 26.}{6.1} \]

\[ = -1.328 \]

The negative Pearson’s measure of skewness shows that the distribution is negatively skewed.

CHAPTER SUMMARY

In this chapter we have looked at various ways of summarising, describing and comparing distributions of data using the following:

- **Measures of Central Tendency**
  - Arithmetic mean, median, mode and geometric mean: definitions, calculations and interpretation

- **Measures of Dispersion**
  - Range, mean deviation, standard deviation, variance and inter quartile range: definitions, calculations and interpretations

- **Coefficient of Variation**
  - Definition, calculation and interpretation
• Measures of Skewness
  Definition of skewness
  Empirical relationship between mean, median and mode
  Pearson’s measure of skewness: Calculation and interpretation

END OF CHAPTER EXERCISES

1. Distinguish between the arithmetic mean, median and the mode of a set of data.

2. An enquiry on 20 families shows the following number of children in the families

<table>
<thead>
<tr>
<th>Number of Children</th>
<th>Number of families</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Calculate the mean, mode and median number of children in the families

3. What is the best “average” if any, to use in each of the following situations? Justify each of your answers.
   a. If a shop sold television sets and wants to find the price of average television set.
   b. To establish a typical wage to be used by an employee in wage negotiations for a small company of 300 employees, a few of whom are very highly paid specialists.
   c. To state the amount to be paid to each employee when the company introducing a profit sharing scheme requires that each employee receives the same amount.
   d. To determine the height of a bridge to be constructed (not a drawbridge) where the distribution of the heights of all ships which would pass under it is known.
   e. To ascertain the average annual income of all workers when it is known that the mean annual income of skilled workers is K900,000 while the mean annual income of unskilled workers is K700,000.00.

4. Consider the following set of measurements which represent the unemployment rates (%) of 12 countries:

7, 9, 11, 5, 3, 5, 10, 12, 5, 6, 7, 4

Find the arithmetic mean, median, mode and geometric mean of these 12 unemployment rates.

5. The following set of data represents a frequency distribution of the unemployment rates (%) of 50 countries:

<table>
<thead>
<tr>
<th>Unemployment rate(%)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 but under 2</td>
<td>5</td>
</tr>
</tbody>
</table>

BUSINESS MATHS & STATISTICS (TC3)
4 but under 6 15
6 but under 8 10
8 but under 10 6
10 but under 12 4

Calculate the mean mode and median of the distribution of the unemployment rate above.

6. Given the following simple data set: 9, 8, 3, 4, 7, 5, 3, find the range, mean deviation, and standard deviation.

7. The following set of data represents the annual acquisition expenses in K million (Km) incurred by 100 insurance companies in 2009:

<table>
<thead>
<tr>
<th>Acquisition Expenses (Km)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 20</td>
<td>10</td>
</tr>
<tr>
<td>20 but less than 40</td>
<td>35</td>
</tr>
<tr>
<td>40 but less than 60</td>
<td>40</td>
</tr>
<tr>
<td>60 but less than 80</td>
<td>10</td>
</tr>
<tr>
<td>80 but less than 100</td>
<td>3</td>
</tr>
<tr>
<td>100 but less than 120</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) Find
i. the arithmetic mean
ii. median
iii. mode for this distribution
iv. the Pearson’s coefficient of skewness and say whether the distribution is positively or negatively skewed.

(b) Calculate the standard deviation of the acquisition expenses.

(c) Calculate the coefficient of variation for the acquisition expenses.

8. Incomes of females in 2006 by highest educational qualification were as follows:

<table>
<thead>
<tr>
<th>Range of Weekly income (K'000)</th>
<th>Degree %</th>
<th>O-level %</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 and under 60</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>60 and under 80</td>
<td>7</td>
<td>30</td>
</tr>
<tr>
<td>80 and under 100</td>
<td>7</td>
<td>28</td>
</tr>
<tr>
<td>100 and under 120</td>
<td>18</td>
<td>14</td>
</tr>
<tr>
<td>120 and under 140</td>
<td>23</td>
<td>7</td>
</tr>
<tr>
<td>140 and under 160</td>
<td>14</td>
<td>3</td>
</tr>
<tr>
<td>160 and under 180</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>180 and under 220</td>
<td>16</td>
<td>1</td>
</tr>
</tbody>
</table>

Given that there were 100 women with degrees and 500 women with O-level
(i) Calculate the mean and standard deviation for weekly income for those with a degree qualification.
(ii) Calculate the mean and standard deviation for weekly income for those with a O-level qualification.
(iii) Using coefficients of variation compare the variability in the distributions.

9. A food processor makes chambo fillet from chambo supplied from the Lake. It is known that fish of weight greater than 1 Kg make good fillets which fetch better prices. The processor further knows that to make a profit, fish weighing at least 1Kg must make up 59% of a consignment. In order to forecast his revenue and therefore profit, he decides to take a sample of 100 fish from different consignments in accordance to a predetermined scientific technique. The following is the frequency distribution of the fish:

Weights of 100 fish in grams:

<table>
<thead>
<tr>
<th>Weight of fish</th>
<th>Number of fish</th>
</tr>
</thead>
<tbody>
<tr>
<td>200 but &lt; 400</td>
<td>2</td>
</tr>
<tr>
<td>400 but &lt; 600</td>
<td>4</td>
</tr>
<tr>
<td>600 but &lt; 800</td>
<td>9</td>
</tr>
<tr>
<td>800 but &lt; 1000</td>
<td>10</td>
</tr>
<tr>
<td>1000 but &lt; 1200</td>
<td>13</td>
</tr>
<tr>
<td>1200 but &lt; 1400</td>
<td>19</td>
</tr>
<tr>
<td>1400 but &lt; 1600</td>
<td>20</td>
</tr>
<tr>
<td>1600 but &lt; 1800</td>
<td>15</td>
</tr>
<tr>
<td>1800 but &lt; 2000</td>
<td>6</td>
</tr>
<tr>
<td>2000</td>
<td>2</td>
</tr>
</tbody>
</table>

Required

a) Draw a histogram of the above distribution
b) Estimate the mean, mode and median fish weight.
c) Which one is the suitable average for the food processor and why?
d) Calculate the standard deviation, and the coefficient of skewness
e) Explain what the statistics calculated in (b) and (c) tell the food processor.
f) Draw an Ogive of the data.
g) Identify the weight range in which the middle 50% of the fish lie.
h) What is the proportion of fish that have weight of 1Kg and above? Advise the food processor on what your calculation shows regarding making a profit.

10. The monthly salaries earned by 30 randomly-selected primary school teachers in a city (to the nearest K’000) are as follows:

<table>
<thead>
<tr>
<th>20</th>
<th>25</th>
<th>26</th>
<th>30</th>
<th>32</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>25</td>
<td>26</td>
<td>30</td>
<td>32</td>
<td>48</td>
</tr>
<tr>
<td>22</td>
<td>25</td>
<td>26</td>
<td>30</td>
<td>36</td>
<td>52</td>
</tr>
<tr>
<td>23</td>
<td>26</td>
<td>28</td>
<td>30</td>
<td>36</td>
<td>60</td>
</tr>
<tr>
<td>25</td>
<td>26</td>
<td>28</td>
<td>32</td>
<td>38</td>
<td>61</td>
</tr>
</tbody>
</table>
(c) Form a simple frequency distribution to represent the distribution of salaries

(d) Find the arithmetic mean, median and mode for the salaries

(e) Calculate the standard deviation.

(f) Calculate the coefficient of skewness and comment on the result.

(g) The 30 teachers have expressed a concern that gaps between their salaries are too big compared to teachers in an adjacent town which has a mean of K20,000 but a lower standard deviation of 8300. Use an appropriate measure to compare the salaries of the two groups of teachers and comment on the concern.
CHAPTER 13 COMBINATORIAL ARITHMETIC

LEARNING OBJECTIVES

By the end of this chapter the student should be able to:

i. Differentiate between permutations and combinations.
ii. Carry out arithmetic computations involving permutations and combinations.
iii. Understand the concepts well enough for use in sample space under probabilities.

13.0 INTRODUCTION

Permutations and Combinations are techniques of quantifying arrangement or selections of items from a group.

13.1 FACTORIALS

13.1.1 Definition

For each positive integer \( n \), the quantity \( n \) factorial, denoted \( n! \), is defined to be the product of all the integers from 1 to \( n \): \( n! = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times (n-1) \times n \). By definition, \( 0! = 1 \).

Example 1

1 factorial or \( 1! = 1 \)
2 factorial or \( 2! = 1 \times 2 = 2 \)
3 factorial or \( 3! = 1 \times 2 \times 3 = 6 \)
7 factorial or \( 7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040 \)

13.1.2 Arithmetic involving factorials

A. Multiplication and division

Factorials can be multiplied and divided in the normal way numbers are multiplied.

Example 2

1) Evaluate a) \( 4! \times 3! \) b) \( 1! \times 0! \times 3! \times 2! \)
2) Find the values of a) \( 4! \div 3! \) b) \( 26! \div 24! \)
   c) \( \frac{n!}{(n-r)!*r!} \), Where \( n = 12 \) \( r = 2 \)
3) Simplify \( \frac{(n+2)!}{(n-1)!} \)
Solution

1  a)  $4! \times 3! = (1 \times 2 \times 3 \times 4) \times (1 \times 2 \times 3) = 24 \times 6 = 144$
    b)  $1! \times 0! \times 3! \times 2! = 1 \times 1 \times 6 \times 2 = 12$

2  a)  $4! \times 3! = \frac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3}$
    b)  $26! \div 24! = 25 \times 26 = 650$
    c)  $\frac{12!}{(12-2)! \times 2!} \times \frac{12!}{(10)! \times 2!}$

3  $(n+2)! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times (n) \times (n+1) \times (n+2)$
   $(n-1)! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1)$
   
   $$\frac{(n+2)!}{(n-1)!} = \frac{1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times (n) \times (n+1) \times (n+2)}{1 \times 2 \times 3 \times 4 \times \ldots \times (n-1)}$$
   $$= n(n+1)(n+2)$$
   $$= n^3 + 3n^2 + 2n$$

B. Addition and subtraction

Addition and subtraction involve the end result or product of the figures being multiply.

Example 3

1  Simplify $5! + 3!$
    $26! \div 24! = 25 \times 26 = 650$

Solution

1  $5! + 3! = 120 + 6 = 126$
    $2! - 3! = 24 - 6 = 18$

Factorials are often used in permutations and combinations

13.2  PERMUTATIONS

13.2.1 General concept

The concept of permutations relates to arranging objects or items in an ordered manner.

Thus a permutation of $r$ objects at a time taken from $n$ is the number of ways $r$ objects can be arranged out of $n$ where the order is important.

Example 4  Given 3 books which on a shelf, find the number of ways in which the three books can be arranged.

Solution  Let the books be those of Algebra, Biology and Chichewa (A, B, C)
If arranged systematically the arrangements are:

- ABC, ACB
- BAC, BCA
- CAB, CBC

There are 6 arrangements (permutations)

**Example 5**
Given the three books in a box, find the number of ways the books taking two at a time can be arranged on the shelf

**Solution**
If the books are ABC the following are the arrangements

<table>
<thead>
<tr>
<th>AB</th>
<th>BA</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>CA</td>
</tr>
<tr>
<td>AD</td>
<td>DA</td>
</tr>
<tr>
<td>BC</td>
<td>CB</td>
</tr>
<tr>
<td>BD</td>
<td>DB</td>
</tr>
<tr>
<td>CD</td>
<td>CD</td>
</tr>
</tbody>
</table>

There 12 arrangements (permutations)

### 13.2.2 Formula

A permutation of say r objects at a time from say n can be calculated using a formula.

A permutations of r objects out of n is denoted by \(^nP_r\) which is also written as \(P_{n,r}\)

\[^nP_r = \frac{n!}{(n-r)!}\]

**Example 6**
Find the number of permutations in

- a) Example 1 and
- b) Example 2 above

**Solution**

a) \(^3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 6/1 = 6\)

b) \(^nP_r = ^nP_2 = \frac{3!}{(3-2)!} = 6\)

**Example 7**
Find the permutations of 2 letters out of 4

**Solution**

\(\frac{4!}{(4-2)!} = \frac{4!}{2!} = 4 \times 3 = 12\)
13.3 COMBINATIONS

13.3.1 Definition
A combination is a selection of r objects out of n where the order is not important.

Example 8  Find the number of way two items can be selected from 4
Solution  Let the item be A, B, C, and D
The arranging these systematically in twos the combinations are:
AB, AC, AD
BC, BD
CD
6 ways

13.3.2 Formula
A combination of say r objects at a time out of n is denoted by

\[ ^nC_r = \frac{n!}{(n-r)! r!} \]

Example 9  Researchers send teams of two data collectors to collect data from a particular town. If they have a total of 6 data collectors, determine the possible number of possible teams than can be selected.

Solution  This is a combination of two collectors at a time out of 6.

\[ ^6C_2 = \frac{6!}{(6-2)! 2!} = \frac{6!}{4! 2!} = \frac{6 x 5}{2} = 15 \text{ possible teams} \]

CHAPTER SUMMARY
This chapter covered
- Permutation (the concept and formula and application)
- Combinations (the concept, formula and application)

STUDENT EXERCISES
1 Calculate  a) 4!  b) 0!  c) 7!
      d) 12!  e) 1!  f) 21!
2 Simplify the following factorial problems:
   a) 3! + 2!
   b) 3! - 2!
   c) 3! \times 2!
d) \( \frac{10!}{3!} \)

e) \( \frac{12!}{(12-4)!} \)

f) \( \frac{59!}{58!} \)

3 Simplify \( ^nP_n \).

4 Express \( ^{10}P_5 \) in terms of factorials and evaluate.

5 Six different books are on a shelf. In how many different ways could you arrange them?

6 How many permutations are there of the letters \( \text{wxyz} \) if four are taken at a time?

7 Evaluate \( ^8C_6 \).

8 Evaluate \( \frac{^7C_4 \times ^6P_2}{^{10}C_3} \)

9 A committee of three is supposed to be selected from a group of 5 people (two ladies and three gentlemen. How many different committees are possible?
LEARNING OBJECTIVES
By the end of the chapter, the student should be able to:

i. Define probability
ii. Describe the role of probability in decision making
iii. Describe the classical, empirical, and subjective approaches to probability
iv. Distinguish experiment, event and outcome
v. Calculate marginal and conditional probabilities
vi. Apply the rules of probability including addition and multiplication rules
vii. Apply a tree diagram to organize and compute probabilities

14.0 INTRODUCTION
The emphasis in the first chapters is on descriptive statistics. Descriptive statistics is concerned with summarizing data collected from past events. Now we turn to a different facet of statistics. This facet allows us to compute the chance that something will occur in future. Seldom does a decision maker have complete information from which to make a decision. In situations like these, one takes a change or some risks to make a decision. Because there is uncertainty in decision making, it is important that all the known risks be statistically evaluated. Helpful in this evaluation and quantification of uncertainty is probability theory, which is broadly referred to as the science of uncertainty. Probability allows a decision maker to analyze the risks and minimize the gamble inherent. Furthermore, probability allows for the determination of reliability and validity of generalisation of sample results to a population.

This chapter introduces the basic probability language, including terminology such as experiment, event, and basic concepts of probability. We also discuss the main rules and principles of probability.

14.1 DEFINITIONS OF KEY TERMS
Probability is the numerical measure of the likelihood that an event in the future will happen. It can only assume a value between 0 and 1 inclusive (i.e. $0 \leq P(A) \leq 1$, where $P(A)$ is the probability of the event $A$).

A value near zero means the event is not likely to happen and a value near one means it is more likely.

Key Terms

a) Experiment:
This term means an undertaking or a process that leads to the occurrence of one and only one of several possible observations. This definition is more general than our understanding of an experiment from physical science. In probability, an experiment has two or more possible results, called outcomes, and it is uncertain which will occur. Examples include tossing a coin, drawing a card, making orders of supplies, hiring out specific vehicles, and exploration/drilling to find oil.

b) Outcome:
An outcome is a particular result of an experiment. For example, if we asked 500 form four students whether they would pursue a career in accounting. We would be uncertain as how many would say ‘yes’. One possible outcome is therefore that ‘321 students indicate that they would pursue a career in accounting’.

185
Another outcome is that ‘210 students would pursue a career in accounting’. Still another possibility is that “all of them would pursue a career in accounting”.

c) Event:
An event is a collection of one or more outcomes of an experiment.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Roll a die</th>
<th>Count the number of companies in liquidation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outcomes</td>
<td>Observe a 1</td>
<td>None in liquidation</td>
</tr>
<tr>
<td></td>
<td>Observe a 2</td>
<td>One in liquidation</td>
</tr>
<tr>
<td></td>
<td>Observe a 3</td>
<td>Two in liquidation</td>
</tr>
<tr>
<td></td>
<td>Observe a 4</td>
<td>…</td>
</tr>
<tr>
<td></td>
<td>Observe a 5</td>
<td>67 in liquidation</td>
</tr>
<tr>
<td></td>
<td>Observe a 6</td>
<td>…</td>
</tr>
<tr>
<td>Some possible events</td>
<td>Observe an even number</td>
<td>More than 21 in liquidation</td>
</tr>
<tr>
<td></td>
<td>Observe a prime number</td>
<td>9 or fewer in liquidation</td>
</tr>
<tr>
<td></td>
<td>Observe an odd number</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Observe a number less than 4</td>
<td></td>
</tr>
</tbody>
</table>

14.2 TYPES OF EVENTS
Considering the definition of an event above it is possible to distinguish two types of event in the context of probability. These are **mutually exclusive event** and **collectively exhaustive**.

Mutually exclusive events are events which do not occur together. The occurrence of one precludes the other. For instance, when a coin is tossed you either get head and tails. These are therefore mutually exclusive. Another example is gender of an individual. One is either male or female.

Figure 14.1 Mutually exclusive events

![Event A](event_a.png) ![Event B](event_b.png)

Mutually exclusive events: No common outcomes

Events that share some common outcomes or can occur simultaneously are said to be no-mutually exclusive or overlapping events.

Figure 14.2 Overlapping events

![Event A](event_a.png) ![Event B](event_b.png)

Common outcomes
Collectively exhaustive events are events from the same sample space whose probabilities add up to 1. If an experiment is conducted, then at least one of these events must occur.

**Sample space**
A sample space is the collection of all possible outcomes of an experiment. When a coin is tossed, the sample space comprises a head and a tail (H, T). Sometimes the sample space is easy to determine, while at other times it is complex. When elements of the sample space are counted, the result is the number of total possibilities.

### 14.3 APPROACHES TO PROBABILITIES
Probabilities are generally of two types based on the approach followed to assigning probabilities. These types are:

- **Subjective**
- **Objective**

#### 14.3.1 Subjective Probabilities
This is the likelihood (probability) of an event occurring that is assigned based on whatever information is available. The source of information can be expert opinion, intuition, or a pure guess. If there is little or no information on which to base a probability, it may be arrived at subjectively. To a larger extent, subjective probability describes a situation where one is simply expressing his or her strength of belief in the way events will turn out.

Some examples of subjective probability

- An ardent supporter of a football team putting the probability of winning the next game high even when the team has lost more games than it has won in the past.
- A candidate estimating the probability that she complete PAEC Technician programme within 18 months.
- A ‘new’ political party estimating the probability of winning the next presidential election.

#### 14.3.2 Objective Probabilities
If a probability can be established scientifically, it is referred to as **objective probability**. Objective probability is subdivided into classical and empirical probability.

**Classical Probability**
This is based on the assumption that the outcomes of an experiment are equally likely. Using the classical approach, the probability of an event is computed by dividing the number of favourable outcomes by the number of possible outcomes i.e.

\[
Probability\ of\ an\ event = \frac{Number\ of\ favourable\ outcomes}{Total\ number\ of\ possible\ outcomes}
\]

In classical probability approach we do not need to carry out an experiment rather we only need to know the nature of the experiment. This is called a priori condition of the experiment. For instance the probability that of obtaining head when a fair coin is tossed is \(\frac{1}{2}\). We do not need to actually toss a coin to find this. All we need is to know that a coin has two faces, head and tail, and are equally likely.
Example 1
Consider an experiment of selecting an accountant from finance department to attend training in a new accounting package. If there are 4 female and 10 male accountants, what is the probability that a female accountant attends the training?

Solution:
We notice that any of the accountants has an equal chance of being selected for the training. Hence the sample space has 14 individual or possibilities

Since we are interested in female accountant, there are 4 favourable outcomes in the collection of 14 equally likely accountants. Therefore:

\[
\text{Probability of female accountant} = \frac{4}{14} = 0.26 \text{ (to 2 dec. Places)}
\]

Empirical Probability
Another way to determine probability is based on relative frequencies. In this approach, the probability of an event occurring is determined by observing what fraction of the time similar events occur in the past i.e.

\[
\text{Probability of an event} = \frac{\text{Number of times event occurred in the past}}{\text{Total number of observations}} = \frac{f}{\sum f}
\]

The empirical approach to probability is based on what is called the law of large numbers which states over large numbers of observations or experiments the empirical probability of an event will approach its true probability.

Example 2
Business person notes that out of 10 orders he made to a supplier only 2 were delivered late. Find the probability that an order he is making now will be late.

Solution:
The probability that the current order will be late is

\[
\frac{\text{Number of times event occurred in the past}}{\text{Total number of observations}} = \frac{2}{10} = \frac{1}{5}
\]

Note
that classical approach has an element of frequency being used. The difference is that under a classical approach the denominator is fixed by the nature of the experiment, while under empirical approach the denominator is not fixed but depends on empirical data from trials.

14.4 MATHEMATICAL DEFINITION OF PROBABILITY
Let \( A \) be an event, then the probability of the event \( A \) occurring, denoted \( P(A) \), is defined mathematically as:

\[
P(A) = \frac{n(A)}{n(S)}
\]

where \( n(A) \) is the number of outcomes in event \( A \), and \( n(S) \) is the number of total possible outcomes.
Example 3
Ziko Ltd is running a promotion involving customers. There are 13 tickets selected from the North, 15 from the Center, 22 from the South and 15 from the Eastern region. These are placed in one drum for a final draw and the draw will be a random process.

a) If there is one grand price, a car, what is the probability the car goes to the Center?
b) Suppose there are two grand prices, where the East and Southern regions compete as an entity while the North and Center compete as another entity, find the probability that:
   i) In East and south entity the car goes to the South and
   ii) In the North and Central entity the car goes to the North

Solution
a) Let \( C \) be the event that the car is won by a customer from the Centre

\[
P(C) = \frac{n(C)}{n(S)} = \frac{15}{13 + 15 + 22 + 15} = \frac{15}{65} = 0.23
\]

b) i) Let \( T \) be the event that the car is won by a customer from the South in the East-South entity

\[
P(T) = \frac{\text{Number of customers from South}}{\text{Number of customers from East and South}}
\]

\[
\therefore P(T) = \frac{22}{15 + 22} = \frac{22}{37} = 0.59
\]

ii) Let \( N \) be the event that the car is won by a customer from the North in the North-Centre entity

\[
P(N) = \frac{\text{Number of customers from North}}{\text{Number of customers from North and Centre}}
\]

\[
\therefore P(N) = \frac{13}{13 + 15} = \frac{13}{28} = 0.46
\]

14.5 BASIC RULES OF PROBABILITY
There are three basic rules probability

i) Probability limits:
The rule states that the probability of any event \( A \) lies between 0 and 1 inclusive

\[
0 \leq P(A) \leq 1
\]

where \( P(A) = 0 \) means event \( A \) cannot occur (i.e. an impossible event) and \( P(A) = 1 \) means event \( A \) is certain to occur.

ii) Total probability rule:
The rule states that the sum of the probabilities of all events/outcomes in a sample simple is 1

\[
\sum_{i=1}^{n} P(A_i) = 1
\]
iii) **Complement rule of probability:**

The rule states that if the probability of an event $A$ occurring is $P(A)$, the probability of the event not occurring is:

$$P(\overline{A}) = 1 - P(A)$$

**Note:** $P(\overline{A}) = P(A') = P(A^c)$

For example, if it is known that 43% of accountants in Malawi are female. We can therefore write;

$P(\text{Female}) = 0.43$, and $P(\text{Male}) = 1 - P(\text{Female}) = 1 - 0.43 = 0.57$

**Example 4**

The following table shows the number of visitors that exited Malawi in 2009.

<table>
<thead>
<tr>
<th>Port of exit</th>
<th>Number of visitors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chileka</td>
<td>76,308</td>
</tr>
<tr>
<td>Chiponde</td>
<td>52,836</td>
</tr>
<tr>
<td>Chitipa/Chisenga</td>
<td>6,816</td>
</tr>
<tr>
<td>Dedza</td>
<td>10,916</td>
</tr>
<tr>
<td>Kaporo/Songwe</td>
<td>90,350</td>
</tr>
<tr>
<td>Lilongwe (KIA)</td>
<td>174,652</td>
</tr>
<tr>
<td>Mchinji/Chimaliro</td>
<td>82,304</td>
</tr>
<tr>
<td>Muloza</td>
<td>41,203</td>
</tr>
<tr>
<td>Mwanza</td>
<td>141,956</td>
</tr>
<tr>
<td>Nayuchi</td>
<td>14,759</td>
</tr>
<tr>
<td>Nsanje/Marka</td>
<td>17,322</td>
</tr>
<tr>
<td>Other</td>
<td>45,609</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>755,031</strong></td>
</tr>
</tbody>
</table>

*Source: 2009 Tourism Report - NSO*

If we randomly select a visitor, what is the probability that the visitor

a) exited through Lilongwe (KIA)

b) did not exit through Chileka

c) exited either through Kaporo, Dedza or Mwanza?

**Solution:**

Let $L =$ visitor exited through Lilongwe, $C =$ visitor exited through Chileka, $K =$ visitor exited through Kaporo, $D =$ visitor exited through Dedza, and $M =$ visitor exited through Mwanza.

a) $P(L) :$

$$P(L) = \frac{n(L)}{n(S)} = \frac{174,652}{755,031} = 0.23$$

b) $P(C) :$
\[ P(\bar{C}) = 1 - P(C) \]
\[ P(C) = \frac{n(C)}{n(S)} = \frac{76308}{755031} = 0.1 \]
\[ \therefore P(\bar{C}) = 1 - 0.1 = 0.9 \]

c) \[ P(K \text{ or } D \text{ or } M) \]
\[ P(K \text{ or } D \text{ or } M) = \frac{n(K) + n(D) + n(M)}{n(S)} = \frac{90350 + 10916 + 141956}{755031} \]
\[ = \frac{243222}{755031} = 0.32 \]

**Intersection of Events**

An intersection of events A and B is the collection of all outcomes that belong to both events A and B. The intersection of events A and b is denoted \((A \cap B)\) i.e. \((A \text{ and } B)\). We illustrate the intersection of event A and B below.

Figure 14.3 Intersection of events

![Intersection of events](image)

\((A \cap B)\)

**Union of Events**

A union of events A and B is the collection of all outcomes that belong to either event A alone or B alone or to both A and B. The union is denoted \((A \cup B)\) i.e. \((A \text{ or } B)\). We illustrate the union of events A and B below:

Figure 14.4 Union of events

![Union of events](image)

\((A \cup B)\)

**Independent Events**

Events are independent if the occurrence of one event does not affect the occurrence of another. They may or may not occur together. For example, person tosses a coin two times. The result on the first toss and of the second are independent, and the height of a person and position held at work are also independent.
Conditional events
In this case, one event is influenced, to a certain extent by a preceding event. For example, rain and clouds in the sky are conditional. Furthermore, profit of a company will be affected or influenced by the level of sales.

Classify the following events as mutually exclusive, independent or conditional,

a) The price of cooking oil rising in shops and floods in Karonga
b) Mr Phiri catching the Axa Coach from Lilongwe to Blantyre and Flying Air Malawi.
c) Being the chief accountant and the height of the person.
d) A consignment arriving late and the same consignment arriving on time from RSA.
e) Age of person and being chief executive.

Solution
a) Independent
b) Mutually exclusive
c) Independent
d) Mutually exclusive

tSome Rule for Computing Probabilities
Now that we have defined probability and looked at most types of events, we turn our attention to calculating the probabilities of two or more events by applying the rules of addition and multiplication.

14.5.1 Addition Rule of Probability (The OR rule)
The rule states that, given events A and B which are mutually exclusive, the probability of their union \((A \cup B)\) written as \(P(A \cup B)\) is given by

\[
P(A \cup B) = P(A) + P(B) \quad \text{or} \quad P(A \text{ or } B) = P(A) + P(B)
\]

This can be referred to as the special rule of addition.

If the events are not mutually exclusive, then the probability of their union (i.e. A or B) is given by:

\[
P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)
\]

where \(P(A \cap B)\) is the probability of the overlap/intersection or the chance that the two occur together.

Note that the addition rule has here been termed the “OR” rule because it is applied where the probability of one event “OR” the other is being sought. The word ‘OR’ suggests that A may occur or B may occur. This also includes the possibility that A and B may occur. The idea of “OR” can be explicit (stated) in the question or it can be implied or deduced from the context.

Example 6
A card is drawn from a pack of 52 playing cards. What is the probability that the card is a 10 or a picture card (Q, J, and K)

Solution
Let A = drawing a 10 and there are 4 10s in the pack and B = drawing a picture card and there are 12 picture cards
P(A) = 4/52 and P(B) = 12/52, A and B are mutually exclusive (we don’t have a picture card that is also a 10)

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{12}{52} = \frac{4}{13} \]

**Example 7**
A card is drawn from a pack of playing cards. Find the probability that it is a ten or a red card.

**Solution**
Let A = drawing a 10, and R = Red card. Note that there are two 10s which are red cards (diamond and red heart). This is the overlap or events happening together.

\[ P(A \cup R) = P(A) + P(R) - P(A \cap R) = \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{7}{13} \]

14.5.2 Multiplication Rule of Probability, The “AND” rule
This rule is used when a probability of two or more events occurring together is required. The condition is that the events must be independent.

The rule states that: Given two events A and B which are independent, the probability of their intersection, \( P(A \cap B) \) is given by:

\[ P(A \cap B) = P(A) \times P(B) \]

**Example 8**
A die is rolled two times. Find the probability of obtaining two 6s.

**Solution**
Let \( A_1 \) and \( A_2 \) be the events of obtaining a 6 on the 1st and 2nd rolling respectively.

\[ P(A_1 \cap A_2) = P(A_1) \times P(A_2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \]

**Example 9**
The Government has advised for consultancy work in the fields of finance and at the same time a new company is also requesting for bids for a financial consultant. Mr. J.J. Kayange (FCCA) weighs his chances of winning the government consultancy to be 0.25 and the new company one to be 0.3. Since the two requests for bids are not related he decides to bid for both. What is the probability that he wins both?

**Solution:**
Let G = winning the government consultancy and C = winning the one from the private company
\[ P(A \cap B) = P(G) \times P(C) = 0.25 \times 0.3 = 0.075 \]

14.6 CONDITIONAL PROBABILITY
A conditional probability is the probability of a particular event occurring, given that another event has occurred. The probability of the event \( A \) given that the event \( B \) has occurred is written \( P(A|B) \) and is given by:

\[ P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{P(A \cap B)}{P(B)} \]
From this definition we can state the general rule of multiplication. Given two events, A and B, the joint probability that both will happen is
\[
P(A \cap B) = P(A | B) \times P(B)
\]
or
\[
P(A \cap B) = P(B | A) \times P(A)
\]

**Example 10**

A card is drawn from a deck of 52 cards. Find the probability that the card is an Ace given it is a red heart.

**Solution**

Let \( A = \) Ace, and \( R = \) Red card.

The probability of an Ace given a red heart is given by dividing the number of Aces of red heart by the total number of red hearts i.e.
\[
P(A | R) = \frac{n(A \cap R)}{n(R)} = \frac{1}{13}
\]

Conditional probability can also be calculated from a contingency table. A contingency table is a two-way table used to classify observations according to two or more identifiable characteristics.

**Example 11**

A sample of 1000 persons screened for a certain disease is distributed according to height and disease status resulting from a clinical examination as follows:

<table>
<thead>
<tr>
<th>Disease Status</th>
<th>None</th>
<th>Mild</th>
<th>Moderate</th>
<th>Severe</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Height</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tall</td>
<td>122</td>
<td>78</td>
<td>139</td>
<td>61</td>
<td>400</td>
</tr>
<tr>
<td>Medium</td>
<td>74</td>
<td>51</td>
<td>90</td>
<td>35</td>
<td>250</td>
</tr>
<tr>
<td>Short</td>
<td>104</td>
<td>71</td>
<td>121</td>
<td>54</td>
<td>350</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>300</td>
<td>200</td>
<td>350</td>
<td>150</td>
<td>1000</td>
</tr>
</tbody>
</table>

A person is chosen at random, find the probability that the person is

a) short
b) in the severe status of the disease
c) short and of severe disease status
d) short given that he/she is of severe disease status

**Solution**

Let \( H \) be the event of being short and \( E \) that of severe disease status.

a) \[
P(H) = \frac{n(H)}{n(S)} = \frac{350}{1000} = 0.35
\]
b) \[
P(E) = \frac{n(E)}{n(S)} = \frac{150}{1000} = 0.15
\]
c) \[
P(H \cap E) = \frac{n(H \cap E)}{n(S)} = \frac{54}{1000} = 0.054
\]
\[ d) \quad P(H \mid E) = \frac{n(H \cap E)}{n(E)} = \frac{54}{150} = 0.36 \]

or

\[ P(H \mid E) = \frac{P(H \cap E)}{P(E)} = \frac{0.054}{0.15} = 0.36 \]

### 14.7 PROBABILITY TREES

A probability tree is a diagram used to show outcomes of an undertaking and their probabilities. The diagram is in form of a tree and the branches represent the outcomes and attached probabilities.

**Example 12**

A coin is tossed two times. Express the outcomes and their probabilities on a tree diagram.

**Solution:**

The tree diagram is shown below:

![Probability Tree Diagram](image)

Note the first throw has two possible outcomes: H or T. Each one has its probability indicated. Given a head on the first throw, there are two further outcomes possible (H or T). Because of the independence of results after tossing a coin, the probabilities are still 0.5 on either of the outcomes. The same is true when the outcome is T on the first toss.

The tree diagram also shows the overall set of possibilities of the undertaking. These are

a) Head on the first toss and Head on the 2nd: \( HH, P(HH) = 0.5 \times 0.5 = 0.25 \)

b) Head on first toss and Tail on second toss: \( HT, P(HT)=0.25 \)

c) Tail on the first toss and Head on the second: \( TH, P(TH) =0.25 \)

d) Tail on the first toss and Tail on the second: \( TT, P(TT)= 0.25 \)

Note that the joint probabilities add up to 1.
Example 13

Maruta travels to South Africa frequently to buy a special chemical used in higher grade chicken feed. Maruta knows that the price of the chemical at his sources can be high if demand is high and low if the demand is equally low. From his past experience he is certain that there is a 0.6 chance that the demand for the chemical is high in RSA. When Maruta sells the chemical at home, he can either make a profit or loss depending on the combination of price he pays in RSA and the demand for chickens. Again using his experience he has calculated that there is a 0.3 probability of making a profit if the cost price in RSA was high and can make a profit with a probability of 0.8 give he bought the chemical at a low price.

Calculate the probability of making a loss

Solution:

We outline the outcomes and their probabilities on a tree diagram.

Let H = high price and L = low price in RSA. Let Pr = the event of making a profit at home, and Lo = making a loss.

P(H) = 0.6, Then P(L) = 0.4 (complementary rule)

The conditional probability P(Pr|H) = 0.3 and P(Lo/H) = 0.7

The conditional Probability P(Pr|L) = 0.8 and P(Lo/L) = 0.2

Figure 12.6 Probability tree

\[
\begin{align*}
\text{RSA: Cost} & \quad \text{Home: Profit} & \quad \text{Events} & \quad \text{Join probabilities} \\
& \quad H & \quad \text{Pr} & \quad 0.3 \\
& \quad L & \quad \text{Lo} & \quad 0.7 \\
& \quad 0.6 & & \\
& \quad 0.4 & \quad \text{Pr} & \quad 0.8 \\
& & \quad \text{Lo} & \quad 0.2 \\
\end{align*}
\]

The probability of making a loss is given by the joint events (H and Lo)or (L and Lo)

\[
P((H \text{ and } Lo) \text{ or } (L \text{ and } Lo)) = P(H \text{ and } Lo) + P(L \text{ and } Lo) \quad \text{Mutually exclusive}
\]

\[
= 0.42 + 0.08 = 0.5
\]
14.8 EXPECTED VALUE

The expected value, sometimes called the mathematical expectation, of a set of values which have associated probabilities of occurrence is calculated as:

\[ E(X) = \sum xP(x) \]

where \( P(x) \) is the probability of a value \( x \). Expected value is therefore the sum of the products of data values and their respective probabilities.

Example 14

During holidays a mathematics student has been fishing in a river close to his village and putting his mathematics to practice by weighing his catch of fish and calculating some probabilities of specific quantities (in weight) he can catch. After a month of fishing he reckons that his catch ranges from 2 Kg to 8 Kg with certain pattern of probabilities which he tabulates as follows:

Table 12.4 Fish weights

<table>
<thead>
<tr>
<th>Weight of fish (kg)</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.075</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
</tr>
<tr>
<td>6</td>
<td>0.175</td>
</tr>
<tr>
<td>7</td>
<td>0.125</td>
</tr>
<tr>
<td>8</td>
<td>0.025</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
</tr>
</tbody>
</table>

Find the expected weight of the catch.

Solution

Let the weight of the catch be represented by \( X \) then,

\[ E(X) = \sum xP(x) \]

\[ = (2 \times 0.075) + (3 \times 0.15) + (4 \times 0.2) + (5 \times 0.25) + (6 \times 0.175) + (7 \times 0.125) + (8 \times 0.025) = 4.775 \text{ kg} \]

Note that an expected value is actually a weighted average.

Example 15

Business man calculates that it is possible to make profits of K2,000,000 if the market conditions are perfect, K800,000 if the market conditions are moderate and a loss of K500,000 if the conditions are poor. The probabilities of perfect, moderate and poor conditions are 0.3, 0.5 and 0.2 respectively. Calculate his expected profit.

Solution

Let \( X = \text{Profit} \)

\[ \therefore E(X) = \sum xP(x) \]

\[ = 0.3 \times 2000000 + 0.5 \times 800000 + 0.2 \times (-500000) = 900000 \]

His expected profit is K900,000.00
CHAPTER SUMMARY

In this chapter we introduced probability as a measure of chance and discussed a number of issues. Specifically we wish to highlight the following:

- The probability values lie between zero and one, i.e. $0 \leq P(A) \leq 1$ where $A$ is an event.
- When they are several possible outcomes of an event and it is impossible for more than one outcome to occur at any one time, then the outcomes are said to be mutually exclusive.
- When two or more events take place, these events are said to be independent when the outcome of one event does not affect the outcome of the other event/s.
- The multiplication rule of probability is applied when all events under consideration must occur. The addition rule of probability is applied when we have an either/or situation.
- A Venn diagram can be used to organize and find probabilities for non-sequential events when the categories or situations overlap. Venn diagrams are usually helpful when number elements are provided rather than probabilities.
- A tree diagram can be used to organise and calculate probabilities when there are sequential events and alternative, non-overlapping situations, each with a different outcome. The situations and outcomes must all have probabilities.

END OF CHAPTER EXERCISES

1. John goes to College each morning by bus and he catches it at 07:10 each morning. Much as the bus sticks to time but of late John has been left stranded at the bus stop on three days in the past 12 days because the bus had been coming full. Tomorrow morning he will attempt to catch the same bus. Calculate John’s probability of catching the bus.

2. A committee of three is supposed to be selected from a group of 5 people (two ladies and three gentlemen).
   a) Find the probability that the committee will comprise ladies only
   b) What is the probability that the committee will be made up of at least 1 lady.

3. Two dice are thrown. What is the probability that
   a) the sum of the scores is greater than 10,
   b) the sum of the scores is greater than 10 given that the first die is 6

4. Over the last few years, an internet company has recorded the number of ‘hits’ to its website at 600,000 and the number of customers who made follow-up enquiries at 40,000, a quarter of whom actually made a purchase. Calculate the probability of a ‘hit’ leading to:
   a) An enquiry
   b) A purchase

5. Students at a certain school were surveyed to find out the mode of transport they used when going to school. The results were:
A student is picked at random,

a) What is the probability that the student comes to school by bus?

b) What is the probability that the student comes to school by car or bus?

6. It is estimated that the probability that an energy saver bulb will last more than 2 is 0.8. If ESCOM installs 500,000 bulbs in one of the towns, estimate the expected number of bulbs that will last more than 2 years.

7. 100 students sat for a particular examination of which 60 were boys. The number of students who passed this examination was 40, of whom 20 were girls. Find the probability of:

a) A student passing the examination
b) A girl passing the examination
c) A selected student who is a boy, failing the examination

8. Goliati Ltd must decide which of two alternative strategies to adopt. The company has asked you to predict the probabilities of different profit and loss levels resulting from the two strategies. The results are shown below:

<table>
<thead>
<tr>
<th>Profit/Loss</th>
<th>Strategy A</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>£1.25 million profit</td>
<td>0.5</td>
<td>0.7</td>
</tr>
<tr>
<td>£250,000 profit</td>
<td>0.4</td>
<td>0</td>
</tr>
<tr>
<td>£250,000 loss</td>
<td>0.1</td>
<td>0.3</td>
</tr>
</tbody>
</table>

a) Calculate the expected value of profit for each strategy.
b) Advise the company on the better strategy to follow.

9. Georgina goes to the library. The probability that she checks out (a) a work of fiction is 0.40, (b) a work of non-fiction is 0.30, and (c) both fiction and non-fiction is 0.20. What is the probability that the student checks out a work of fiction, non-fiction, or both?

10. In a library box, there are 8 Accounting, 8 Mathematics, and 8 Communication books. If Chisomo selects two books at random, what is the probability of selecting two different kinds of books in a row?

11. What is the probability of drawing an ace from a standard deck of cards, given that the card is a diamond?

12. A survey of Chimasula Primary School senior (Standard 6, 7 and 8) students asked the question: What is your favorite sport? The results are summarized below:
<table>
<thead>
<tr>
<th>Class</th>
<th>Football</th>
<th>Netball</th>
<th>Volley ball</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>68</td>
<td>41</td>
<td>46</td>
<td>155</td>
</tr>
<tr>
<td>7</td>
<td>84</td>
<td>56</td>
<td>70</td>
<td>210</td>
</tr>
<tr>
<td>8</td>
<td>59</td>
<td>74</td>
<td>47</td>
<td>180</td>
</tr>
<tr>
<td>TOTAL</td>
<td>211</td>
<td>171</td>
<td>163</td>
<td>545</td>
</tr>
</tbody>
</table>

Using these 545 students as the sample space, a student from this study is randomly selected.

a) What is the probability of selecting a student whose favorite sport is volley ball?
b) What is the probability of selecting a Standard 6 student?
c) What is the probability of selecting a student who likes netball and is in standard 8?
d) What is the probability of selecting a student who likes volley ball or a standard 7 student?
e) If the student selected is in 7, what is the probability that the student prefers Netball?
f) If the student selected prefers Football, what is the probability that the student is in standard 6?
g) If the student selected is in standard 8, what is the probability that the student prefers Football?
LEARNING OBJECTIVES
At the end of this chapter, students should be able to:

i. explain the meaning of regression analysis
ii. identify practical examples where regression analysis can be used
iii. plot scatter diagrams
iv. construct a simple linear regression model
v. prepare estimates of the unknown variable using the regression model
vi. compute and interpret the correlation coefficient
vii. compute and interpret the coefficient of determination
viii. create an understanding of linear regression

15.0 INTRODUCTION
In practice many entities which are measured appear to be related to or they influence each other. Easy examples of such situations are given below:

- Advertising expenditure is assumed to have an influence on sales volumes.
- Share price is influenced mainly by a company’s return on investment.
- Hours of operator training is likely to impact positively on productivity.
- Operating speed of a bottling machine affects the reject rate of under-filled bottles.

Because the entities change in terms of quantum, they are referred to as variables. Relationships between variables need to be investigated in terms of existence, nature and strength of the relationship (association). Asituation broadly known as correlation. One reason for such analyses is that in many business decisions, it is necessary to predict the unknown values of a numeric variable using other numeric variables for which values are known.

Techniques for establishing and measuring the strength of the relationship include use of a scatter diagram and calculation of what is termed a correlation coefficient. Prediction is carried out by using among other techniques, regression analysis. The validity and dependability of predictions are dependent on the strength of correlation.

Regression and correlation analysis require that the data type for all variables e.g. marketing, economic, financial, production, human resources, etc.) must be numeric.

This chapter looks and regression analysis first then considers correlation analysis. Specifically it looks at the following topics:

- Simple linear regression analysis, which examines the relationship between two numeric variables only;
- Correlation analysis, which computes the strength of a relationship.

Note: It does not consider, multiple linear regression analysis, where numerous numeric measures are used to influence the outcome of a single numeric measure.
15.1 REGRESSION ANALYSIS

A **scatter plot** of pairs of data between two numeric random variables, \( x \) and \( y \), visually displays the relationship between the two variables, as illustrated graphically in Figure 9.1.

A scatter plot between pairs of \( x \) and \( y \) data values

**Figure 15.1 Advertising expenditure v Sales volume**

<table>
<thead>
<tr>
<th>Sales Vol x ('000)</th>
<th>Ads Exp y ('000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>16</td>
</tr>
<tr>
<td>10</td>
<td>23</td>
</tr>
<tr>
<td>12</td>
<td>21</td>
</tr>
<tr>
<td>13</td>
<td>25</td>
</tr>
</tbody>
</table>

**Correlation analysis** measures the strength of this identified association between the variables.

If a structural (mathematical) relationship exists between two numeric random variables – and can be measured and quantified - then knowing the values of one of the variables, \( x \), can be used to predict (or estimate) the outcome of the other variable, \( y \), for which values are generally unknown. This is the primary purpose of regression and correlation analysis. These techniques can provide managers with a powerful tool for prediction purposes.

**Regression analysis** is a statistical process for estimating the mathematical relationships among numeric random variables as a mathematical equation (usually a straight-line equation). The relat

In Figure 15.1, the straight-line equation would be fitted using regression analysis. The degree of closeness of the plots to the straight line is measured by correlation analysis. The straight equation can be used to estimate the \( y \)-values based on known \( x \)-values. Correlation analysis provided a measure of the confidence a manager can have in the estimated \( y \)-values.

15.2 SIMPLE LINEAR REGRESSION ANALYSIS

Simple linear regression analysis finds a straight-line equation between the values of two numeric random variables only. The one variable is called the **independent** or **predictor variable**, \( x \), and the other is termed the **dependent** or **response variable**, \( y \).
15.2.1 Independent variable ($x$)
The independent variable is represented by the symbol $x$. It is the variable influencing the outcome of the other variable. For this reason it is also called the predictor variable. Its values are usually known or easily determined. In certain instances, the independent variable's values can be controlled or manipulated. In the examples in section 9.1 above, the independent variables would be: advertising expenditure; company return on investment, hours of operator training; and bottling machine speed.

15.2.2 Dependent variable ($y$)
The other numeric random variable is called the dependent variable and is represented by the symbol $y$. The dependent variable is influenced by (or responds to) the independent variable. Hence, it is also called the response variable. Values for the dependent variable are not readily known and need to be estimated from values of the independent variable ($x$). In the examples in section 9.1 above, the dependent variable are: sales volumes; share price; productivity; and reject rate.

In simple linear regression, only one independent variable, $x$, is used to estimate or predict values of the dependent variable, $y$ unlike in multiple regression where two or more independent variables are used to estimate the value of the dependent variable.

To build a simple linear regression mode, a number of steps are followed, as illustrated below.

**Identify the Dependent and Independent Variables**
An essential prerequisite is to correctly identify the independent and dependent variables. This is necessary to ensure that a valid relationship is established. A useful rule of thumb is to ask the following question: “Which variable is to be estimated?”

The answer to this question will identify the dependent variable, $y$. Thus the logic of relationship must be checked before proceeding with regression analysis.

**Use a Scatter Plot to Graphically Examine the Relationship Between the Dependent and Independent Variables**
Consider the following data on volume of sales and advertising expenditure.

The first step towards identifying a possible relationship between two numeric random variables is to prepare a visual plot of their data values. This is done through a scatter plot or scatter diagram.

For the data on advertising expenditure and sales volume the scatter plot is shown along side the data in figure 15.1.

A scatter plot (or scatter graph) graphically displays all pairs of data values of the independent and dependent variables on an $x – y$ axis. The $x$ values are recorded along the horizontal axis and the $y$ values along the vertical axis, as was shown in Figure 15.1.

A visual inspection of the scatter plot will show whether there is a relationship between the two variables, $x$ and $y$, and how strong it is likely to be. The scatter in Figure 15.1 show an upward pattern. As $x$ is rising so is $y$. There is likely to be a strong relationship between the two variables. These initial insights are likely to be reflected in the regression and correlation analysis findings.
If for example, the data points are widely scattered, then a linear regression equation will be of little value in estimating the $y$-variable. The correlation measure will also show almost no association.

Figures 15.2 and 15.3 show various possible patterns of relationships between a dependent numeric variable, $y$, and an independent numeric random variable, $x$.

No linear relationship exists if values of $x$ and $y$ are randomly scattered (i.e. for any given $x$ value, $y$ can have any value over a wide range). This is the case in Figure 15.2. On the other hand, we observe Inverse linear relationship with small dispersion (i.e. for any given $x$ value, the range in $y$ values is small) in Figure 15.3.

From a manager’s perspective, the pattern shown in Figures 15.1 and 15.3 are the most desirable as they show strong linear relationships between $x$ and $y$. Estimate of $y$ based on these relationships will be highly reliable. However, the pattern shown in Figure 15.2 is evidence of no statistical relationship between the two numeric measures. In such cases, there is no value in using regression analysis to estimate $y$ based on $x$ values. The estimates will be unreliable.

### 15.2.3 Calculating the Linear Regression Equation

Regression analysis finds the equation of the best-fitting straight line to represent the actual data points.

**Formula**

A straight line graph is defined as follows

$$\hat{y} = a + b \cdot x$$

Where $x$ = values of the independent variable  
$\hat{y}$ = estimated values of the dependent variable

$a$ = the $y$ intercept coefficient (where the regression line cuts the $y$ axis)  
$b$ = the slope (gradient) coefficient of the regression line  
(i.e. for every one unit change in $x$, $y$ will change by $b$)
15.2.4 Techniques for estimating the regression equation

We will consider three techniques for estimating the regression equation. These are:

a) Scatter graph and line of best fit  
b) The high low method  
c) The least squares method

Example 1

XYZ Ltd manufactures rubber shoe soles often bought by retailers and who then resale them to shoe repairers in town. XYZ can make any order size in thousands depending on the customers’ needs.

The managing director wants a simple model he can use to predict the cost of any order made by a customer before that order is worked on. For this he has extracted records on the last 10 orders made and delivered. The data is as follows:

Table 15.1 Order sizes and cost

<table>
<thead>
<tr>
<th>Order number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order size '000 (x)</td>
<td>5</td>
<td>10</td>
<td>50</td>
<td>15</td>
<td>2</td>
<td>30</td>
<td>60</td>
<td>30</td>
<td>3</td>
<td>50</td>
</tr>
<tr>
<td>Cost '000 (y)</td>
<td>190</td>
<td>240</td>
<td>350</td>
<td>250</td>
<td>300</td>
<td>310</td>
<td>395</td>
<td>335</td>
<td>300</td>
<td>300</td>
</tr>
</tbody>
</table>

Estimate the regression line using

a) Scatter graph and line of best fit  
b) The high low method  
c) The least squares method

Solution

a) Using the scatter diagram.

The scatter diagram appears as follows:
The regression equation is

\[ y = a + bx \]  \hspace{1cm} \text{(in general)}

Where \( a \) = y intercept i.e. value of y when \( x = 0 \)
\[ a = 245 \text{ from the line of best fit} \]

\( b \) = slope and can be determined by picking any two points on the graph

Let the points be \((x_1, y_1)\) and \((x_2, y_2)\)

\[ b = \frac{y_2 - y_1}{x_2 - x_1} \text{ i.e. } \frac{\text{change in } y}{\text{change in } x} \]

Using the graph let \((x_1, y_1) = (0, 245)\) and \((x_2, y_2) = (50, 345)\)

\[ b = \frac{345 - 245}{50 - 0} = 2 \]

The regression equation is \( y = 245 + 2x \)

b) Using the high low method:

\[ y = \text{low } y + \frac{\text{high } y - \text{low } y}{\text{high } x - \text{low } x} \cdot (x - \text{low } x) \]

\[ y = 245 + \frac{345 - 245}{50 - 0} \cdot (x - 0) \]

\[ y = 245 + 2x \]
The constant “a” can be determined by substituting values of one of the coordinates and b into the general linear equation and solving for it.

Generally \( y = a + bx \)

Using the “low” coordinate \((5, 190)\) we have

\[
190 = a + 3.73 \times 5
\]

\[
A = 190 - 18.65 = 171.35
\]

The regression equation is \( y = 171.35 + 3.73x \)

c) The method of least squares (MLS)

Regression analysis uses the method of least squares to find the best-fitting straight-line equation to the plotted data points. The method of least squares is a mathematical technique which finds values for the coefficients, \(a\) and \(b\), such that:

"the sum of the squared deviations of the data points from the fitted line is minimized."

A brief explanation of the rationale is that it considers the vertical deviations between the actual values \(y_i\) and the estimated values \(\hat{y}_i\).

i. A deviation (error) (written as \(e_i\), which is a measure of the vertical distance from an actual \(y\)-value to the fitted line, is first computed for each \(y_i\)-value.

\[
e_i = (y_i - \hat{y}_i)
\]

ii. Each deviation is now squared to avoid positive and negative deviations canceling each other out when summed.

\[
e_i^2 = (y_i - \hat{y}_i)^2
\]

iii. A measure of total squared deviations is then found by summing the individual squared deviations.

\[
\sum e_i^2 = \sum (y_i - \hat{y}_i)^2
\]
iv. Values for \( a \) and \( b \) are now found, which will minimize the sum of these squared deviations in (iii). The mathematical calculation that will minimize the sum of these squared deviations is called the **method of least squares**.

Without showing the mathematical calculations, the coefficients \( a \) and \( b \) that result from the method of least squares are given as follows.

\[
b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}
\]

\[
a = \frac{\sum y - b \sum x}{n} \quad \text{or} \quad a = y - b \frac{x}{n}
\]

The values of \( a \) and \( b \) that are found from the above formulae define the best-fitting linear regression line. This means that no other straight-line equation can be found that will give a better fit (i.e. a smaller sum of squared deviations) than the regression line.

In this example,

**Table 15.2 Calculation table (order sizes and cost)**

<table>
<thead>
<tr>
<th>Order number</th>
<th>Order size '000 (x)</th>
<th>Cost MK'000 (y)</th>
<th>xy</th>
<th>x²</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>190</td>
<td>950</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>240</td>
<td>2400</td>
<td>100</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>350</td>
<td>17500</td>
<td>2500</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>250</td>
<td>3750</td>
<td>225</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>300</td>
<td>600</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>310</td>
<td>9300</td>
<td>900</td>
</tr>
<tr>
<td>7</td>
<td>60</td>
<td>395</td>
<td>23700</td>
<td>3600</td>
</tr>
<tr>
<td>8</td>
<td>30</td>
<td>335</td>
<td>10050</td>
<td>900</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>300</td>
<td>900</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>50</td>
<td>300</td>
<td>15000</td>
<td>2500</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>255</td>
<td>2970</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>84150</td>
<td>10763</td>
</tr>
</tbody>
</table>

\( n = 10 \) (pairs of observations)

Now

\[
b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}
\]

\[
= \frac{10 \times 84150 - 255 \times 2970}{10 \times 10763 - 255^2} = 1.98
\]
The regression equation is \( y = 246.63 + 1.98x \)

15.2.5 Estimating \( y \) – values using the Regression Equation

The regression equation can now be used to estimate values of \( y \) from (known) \( x \)-values. Estimates of \( y \) are found by substituting a given \( x \)-value into the regression equation. The values of \( x \) that can be substituted in the regression equation should lie only within the domain of the \( x \) variable. The domain of the \( x \) variable is defined as the range of \( x \)-values from the data set that were used to calculate the regression line.

15.2.6 The dangers of extrapolation

Extrapolation occurs when \( y \)-values are estimated using \( x \)-values that lie outside the domain of the \( x \)-values. Valid estimates of \( y \) are produced only from \( x \)-values that lie within its domain. If the values of \( y \) are estimated for \( x \)-values outside the limits of the domain (i.e. extrapolation has taken place), the estimates may be invalid, as the relationship between \( x \) and \( y \) beyond these limits is unknown (or has not been defined). The relationship may in fact be quite different from that which is defined between \( x \) and \( y \) within the \( x \)-domain. Extrapolation can sometimes lead to absurd and meaningless estimates of \( y \). The following example illustrates the above process of building a simple linear regression model.

Example 2

The least squares regression equation of cost against order size in the example above is:

\[ y = 246.63 + 1.98x \quad \text{where } x = \text{order size}, y = \text{Cost} \]

Estimate the costs of order sizes of

i) 60,000

\[ y = 246.63 + 1.98 \times 60 = 364.43 \quad \text{or} \quad K364,430 \]

A predicted value will usually be different from an actual observation because the former is a result of the formula.

ii) 55,000

\[ y = 246.63 + 1.98 \times 55 \]
15.2.7 Validity and reliability of a prediction

a. Validity

The equation and therefore any predicted variables are valid only within the range of the original observations.

The reason is that a regression equation is calculated using a set of observations which have a range. The mathematical relationship may not hold true outside the range of the data used. Another way of looking at the same thing is a linear regression is described as such because the relationship between the variables is linear. This is true for the data at hand. The relationship may not be linear outside the range of the given data.

Reliability and accuracy of a prediction.

Dependability and accuracy of a prediction is defined by the extent of correlation between the two variables. The higher the correlation the more reliable and therefore accurate the predicted value is.

Thus a higher correlation coefficient (say $r = 0.9$) is indicative of good relationship and therefore the fact that a prediction is on a sound basis.

Reliability and accuracy are also strengthened by the coefficient of determination. The coefficient of determination indicates the percentage of the dependent variable which is explained by the other (the independent variable). In other words, it shows the degree to which the variables vary together.

The higher this proportion the better the accuracy in prediction.

15.3 CORRELATION ANALYSIS

The reliability of the estimate of $y$ is determined by the strength of the relationship between the $x$ and the $y$ variables. A strong relationship will result in a more accurate and reliable estimate of $y$.

**Correlation analysis** measures the strength of the linear association between dependent and independent $x$ and $y$.

Examples

a) Revenue and profit levels are related. The higher the revenues, the higher the profits will be.
b) height and weight of people are related; taller people tend to be heavier than shorter people.
c) The time it takes to cover a distance by car and the speed of the car.

15.3.1 Techniques for determining correlation

There are several techniques and measures to determine correlation. The most common ones include

a) Inspection of a scatter diagram
b) The Pearson’s product moment correlation coefficient
c) The spearman’s rank correlation coefficient

We will look at each technique on the following sections.
A. Scatter diagram technique

This involves plotting a scatter graph. The pattern in the points of the scatter graph would reveal a pattern in the data that may suggest the degree of the relationship between the variables.

Example 3

The manager of Tikwere Ltd has been wondering whether or not there is a relationship between turnover and profit before tax. To confirm this he has gone to past records and extracted turnover and profit figures as follows but he does not know how to use them to show the correlation:

Table 15.3 Turn over and profit before tax

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn over (millions)</td>
<td>106</td>
<td>125</td>
<td>147</td>
<td>167</td>
<td>187</td>
<td>220</td>
</tr>
<tr>
<td>Profit before tax (millions)</td>
<td>10</td>
<td>12</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>22</td>
</tr>
</tbody>
</table>

Draw a scatter diagram and advise the manager on what it reveals

Solution

a) Figure 15.5 The scatter diagram (turnover and profit before tax)

The scatter of point shows a pattern (upwards). This indicates that there is correlation between the two variables.

B. The Pearson’s Product Moment Correlation Coefficient (r)

The Pearson’s Product moment Correlation coefficient indicates whether or not there is correlation but it also gives the strength of that relationship.
If the correlation coefficient is constructed using the full population data of $x$ and $y$, it is represented by the symbol $\rho$ (rho). If only sample data was used to compute the correlation coefficient, then the sample correlation coefficient is represented by the symbol $r$. In practice, only a sample correlation coefficient is usually computed, hence the term $r$ is commonly used.

**Interpretation of correlation coefficient**
A correlation coefficient is a proportion that takes on values between $-1$ and $+1$ only.

\[-1 \leq r \leq +1\]

a) A value of $r = 1$ means perfect correlation
b) $0 < r < 1$ means positive partial (as opposed to perfect) correlation
c) $r = 0$ stands for no correlation
d) $-1 < r < 0$ means negative partial correlation
e) When $r = -1$, its perfect but negative correlation

Any interpretation should take the following two points into account:

i. A low correlation does not necessarily imply that the variables are unrelated, but simply that the relationship is poorly described by a straight line. A non-linear relationship may well exist. The correlation coefficients that we consider here do not measure non-linear relationships.

ii. A correlation does not imply a cause and effect relationship. It is merely an observed statistical association.

C. Computation of Pearson’s Correlation Coefficient

Pearson’s coefficient represents the correlation between two numerical random variables only and is computed as follows:

**Formula**

\[
r = \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{[n \sum x^2 - (\sum x^2)^2][n \sum y^2 - (\sum y)^2]}}
\]

Where $r$ = the sample Pearson’s correlation coefficient

$x$ = the values of the independent variable

$y$ = the values of the dependent variable

$n$ = the number of paired data points in the sample

Pearson’s correlation coefficient formula is derived from the least squares regression approach, hence its formula has similar terms to the regression coefficients.

**Example 4**

Consider the turnover and profit figures for Tikwere Ltd. Calculate the product moment correlation Coefficient and interpret the result.
Solution

Table 15.4   Turn over and profit (calculation table)

<table>
<thead>
<tr>
<th>Year</th>
<th>T/O</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>2005</td>
<td>106</td>
<td>10</td>
</tr>
<tr>
<td>2006</td>
<td>125</td>
<td>12</td>
</tr>
<tr>
<td>2007</td>
<td>147</td>
<td>16</td>
</tr>
<tr>
<td>2008</td>
<td>167</td>
<td>17</td>
</tr>
<tr>
<td>2009</td>
<td>187</td>
<td>18</td>
</tr>
<tr>
<td>2010</td>
<td>220</td>
<td>22</td>
</tr>
<tr>
<td>Totals</td>
<td>952</td>
<td>95</td>
</tr>
</tbody>
</table>

Now \( n = 10 \), \( \sum x = 952 \), \( \sum xy = 15957 \), \( \sum x^2 = 159728 \), \( \sum y^2 = 1597 \), \( \sum y = 95 \)

So \( r = \frac{10 \times 15957 - 952 \times 95}{\sqrt{[10 \times 159728 - 952^2] \times [10 \times 1597 - 95^2]}} \approx 0.98 \)

A correlation coefficient of \( r = 0.98 \) indicates a very strong correlation between the variables.

Note: Due to the relationship among scatter plots, linear regression and correlation, it is normal to consider them together as the following example illustrates.

Example 5

In the following set of data, \( y \) represents the number of annual claims for flood damage received by an insurance company (in thousands) and \( x \) represents the annual rainfall (in centimeters) over a period of 10 years.

Table 15.5  Flood damage claims and rainfall

<table>
<thead>
<tr>
<th>( y ) (000s)</th>
<th>( x ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>210</td>
</tr>
<tr>
<td>0.5</td>
<td>150</td>
</tr>
<tr>
<td>0.2</td>
<td>120</td>
</tr>
<tr>
<td>4.0</td>
<td>450</td>
</tr>
<tr>
<td>4.0</td>
<td>400</td>
</tr>
<tr>
<td>2.5</td>
<td>200</td>
</tr>
<tr>
<td>2.0</td>
<td>210</td>
</tr>
<tr>
<td>4.0</td>
<td>430</td>
</tr>
<tr>
<td>0.1</td>
<td>190</td>
</tr>
<tr>
<td>4.0</td>
<td>400</td>
</tr>
</tbody>
</table>

(a) Plot the data on a scatter diagram and comment on the likely relationship between \( x \) and \( y \).
(b) Find the equation of the least-squares regression line, assuming that insurance claims for flood damage depend on the amount of rainfall.

(c) Calculate the correlation coefficient and comment on the result.

(d) Use your results to predict the number of flood damage claims in years with 50cm of rainfall and in years with 250cm of rainfall. Comment on the likely validity and accuracy of your predictions.

Solution

a) Figure 15.6 Scatter diagram

![Scatter diagram](image)

Much as points appear in clusters on the scatter diagram, there is a general upward trend which shows correlation. There is a pattern which indicates correlation.

Table 15.6 Workings:

<table>
<thead>
<tr>
<th>( x(cm) )</th>
<th>( y(K'm) )</th>
<th>xy</th>
<th>( x^2 )</th>
<th>( y^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>210</td>
<td>2</td>
<td>420</td>
<td>44100</td>
<td>4</td>
</tr>
<tr>
<td>150</td>
<td>0.5</td>
<td>75</td>
<td>22500</td>
<td>0.25</td>
</tr>
<tr>
<td>120</td>
<td>0.2</td>
<td>24</td>
<td>14400</td>
<td>0.04</td>
</tr>
<tr>
<td>450</td>
<td>4</td>
<td>1800</td>
<td>202500</td>
<td>16</td>
</tr>
<tr>
<td>400</td>
<td>4</td>
<td>1600</td>
<td>160000</td>
<td>16</td>
</tr>
<tr>
<td>200</td>
<td>2.5</td>
<td>500</td>
<td>40000</td>
<td>6.25</td>
</tr>
<tr>
<td>210</td>
<td>2</td>
<td>420</td>
<td>44100</td>
<td>4</td>
</tr>
<tr>
<td>430</td>
<td>4</td>
<td>1720</td>
<td>184900</td>
<td>16</td>
</tr>
<tr>
<td>190</td>
<td>0.1</td>
<td>19</td>
<td>36100</td>
<td>0.01</td>
</tr>
<tr>
<td>400</td>
<td>4</td>
<td>1600</td>
<td>160000</td>
<td>16</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>2760</strong></td>
<td><strong>23.3</strong></td>
<td><strong>8178</strong></td>
<td><strong>908600</strong></td>
</tr>
</tbody>
</table>

\( n = 10 \)
Now \[ b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2} \]
\[ = \frac{10 \times 8178 - 2760 \times 23.3}{10 \times 908600 - 2760^2} = 0.119 \]

and \[ a = \frac{\sum y}{n} - b \frac{\sum x}{n} = \frac{23.3}{10} - 0.119 \times \frac{2760}{10} = -0.95 \]

The regression equation: \[ y = -0.95 + 0.119x \]

c) The correlation coefficient
\[ r = \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{n \sum x^2 - (\sum x)^2} \times \sqrt{n \sum y^2 - (\sum y)^2}} \]
\[ = \frac{10 \times 8178 - 2760 \times 23.3}{\sqrt{10 \times 908600 - 2760^2} \times \sqrt{10 \times 78.5 - 23.3^2}} = 0.926 \]

The coefficient of determination \[ r^2 = (-0.926)^2 = 0.85 \]

Both the coefficients are high. There is strong correlation between rainfall and claims while 85% of claims are explained by the rainfall.

d) When \( x = 50 \)
\[ y = -0.95 + 0.119 \times 50 \]
\[ y = 5 \]
This prediction may be accurate because of high \( r \) but not valid because 50 is outside the range of observed data

When \( x = 250 \)
\[ y = -0.95 + 0.119 \times 250 \]
\[ y = 28.8 \]
The prediction is accurate and valid.

There is high coefficient of determination and 250 is in the range of the observations.
A graphical presentation of the interpretation of values of the Correlation Coefficient

**Figure 15.7 Scatter diagram and associated values of correlation coefficient**

$r = +1$ perfect positive correlation
Scatter of points fall on a straight line rising together.

$r = 0$ Scatter of points does not show any pattern

$r = -1$ Negative, perfect correlation
Points fall on a straight line but variables vary in opposite directions.

**Measure of the strength of correlation**

Perfect correlation (indicated by values of $r = 1$ or -1) is the strongest level a relationship (correlation) between variables can be.

The other extreme is where there is no correlation indicated by $r = 0$

Ranges of $r$ between 0 and 1 indicate positive correlation while ranges of $r$ between -1 and 0 indicate negative correlation. The closer $r$ is to 1 or -1, the stronger the measure of correlation.
Generally $r = 0.5$, for positive correlation and -0.5 for negative correlation are taken to be cut off points.

The following diagram illustrates the measures:

**Figure 15.8  Strengths of correlation**

![Figure 15.8  Strengths of correlation](image)

**Example 6**

Describe the strengths of the following coefficients of correlation calculated on various sets of data.

a) $r = -0.98$

b) $r = 0.8$

c) $r = 0.4$

d) $r = -0.6$

e) $r = -0.2$

**Solution**

a) $r = -0.98$  correlation is very strong

b) $r = 0.8$  correlation is strong

c) $r = 0.4$  this is weak correlation

d) $r = -0.6$  fairly strong correlation

e) $r = -0.2$  weak correlation

**15.3.2 Coefficient of Determination**

When the sample correlation coefficient, $r$, is squared ($r^2$), the resultant statistical measure is called the **coefficient of determination**, denoted $R^2$.

The coefficient of determination, $r^2$, defined as the proportion (or percentage) of variation in the dependent variable, $y$, that is explained by the independent variable, $x$.

The coefficient of determination ranges between 0 and 1 (or 0% and 100%)

i.e. $0 \leq r^2 \leq 1$

**To interpret the coefficient of determination**

The proportion (or percentage) of variation $y$ that $x$ can explain is a measure of how strongly $x$ and $y$ are associated. If $x$ can explain a high proportion (or percentage) of the variation in $y$, then $x$ and $y$ are strongly associated and vice versa.
When \( r^2 = 0 \) no variation in \( y \) can be explained by the \( x \) variable. This corresponds to the scatter plot in which \( r = 0 \), where \( x \) is of no value in estimating \( y \). There is no association between \( x \) and \( y \).

When \( r^2 = 1 \) the values of \( y \) are completely explained by the \( x \)-values. There is perfect association between \( x \) and \( y \). This is where \( x \)-values exactly estimate the \( y \)-values.

When \( 0 < r < 1 \)

- Values of \( r^2 \) that lie closer to zero (or 0%) indicate a low percentage of variation in \( y \) explained by the \( x \) variable. This represents a weak association between \( x \) and \( y \).
- Alternatively, values of \( r^2 \) that lie closer to 1 (or 100%) show that the \( x \) variable is of real value in estimating the actual values of the \( y \) variable. This represents a strong association between \( x \) and \( y \).

**Example 7**

Consider the following situations

<table>
<thead>
<tr>
<th>Kgs of sugar purchased</th>
<th>Cost (MK)</th>
</tr>
</thead>
</table>

**Table 15.7**

<table>
<thead>
<tr>
<th>Situation A: Sugar Purchases</th>
<th>Situation B: Time fishing &amp; Catch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kgs</td>
<td>Cost (MK)</td>
</tr>
<tr>
<td>-----</td>
<td>----------</td>
</tr>
<tr>
<td>1</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>400</td>
</tr>
<tr>
<td>3</td>
<td>600</td>
</tr>
<tr>
<td>4</td>
<td>800</td>
</tr>
<tr>
<td>5</td>
<td>1000</td>
</tr>
<tr>
<td>6</td>
<td>1200</td>
</tr>
</tbody>
</table>

For each situation:

a) plot the scatter diagrams
b) calculate the coefficients of determination using the correlation coefficients shown and interpret the coefficients of determination.
Solution

a) Figure 15.9 Scatter diagrams:

<table>
<thead>
<tr>
<th>Situation A:</th>
<th>Situation B:</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Goods</th>
<th>Purchased Cost</th>
<th>Time (Hrs)</th>
<th>Quantity of Sugar (Kgs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fishing Catch</td>
<td>1</td>
<td>0.5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>5</td>
<td>14</td>
</tr>
</tbody>
</table>

Correlation coefficient $r = 1$

$R^2 = 1^2 = 1$

All increases in the cost are accounted for or explained by increases in the quantity of sugar.

b) Coefficients of determination:

Situation A:

$r = 1$:
It is perfect correlation. The scatter-graph shows perfect goodness of fit in that all point lie on the straight line

The coefficient of determination $R^2 = 1^2 = 1$
All increases in the cost are accounted for or explained by increases in the quantity of sugar.

Situation B

$r = 0.83$:
Correlation is partial and the scatter diagram shows an imperfect fit

$r^2 = 0.63$ meaning only 63% of variations in fish caught are explained by variation in time spent fishing.

The scatter diagram also shows that there are other factors explaining the catch in addition to time. For example 3 hours resulted in 8 kgs while 4 hours had a catch of 5 kgs.

Example 8

An economist has put it that the rate of unemployment is related to its Gross Domestic Product (GDP). In a bid to prove this an economist collects data on GDP and associated rates of unemployment for the past 10 years where the rate of inflation has been relatively stable. The figures are as follows:
Table 15.8 GDP and rate of unemployment

<table>
<thead>
<tr>
<th>GDP</th>
<th>Rate of Unemployment</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>9</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>19</td>
<td>5</td>
</tr>
<tr>
<td>22</td>
<td>3</td>
</tr>
<tr>
<td>22</td>
<td>3</td>
</tr>
</tbody>
</table>

Required.

a) Taking X to represent GDP, and Y to represent unemployment plot a scatter diagram and comment on the likely relationship between GDP and unemployment.

b) Calculate the Pearson correlation coefficient of the data and comment on the result.

c) Calculated the coefficient of determination and comment on its likely use.

Solution

a) Figure 15.5 The scatter diagram

The scatter of points produce a pattern which is slopping downwards to the right signifying that unemployment is negatively related to GDP.

Further the points are not on a straight line meaning the correlation is not perfect but partial. However since the points a close to forming a straight line, the correlation is strong.
b) The correlation coefficient

\[ r = \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{n \sum x^2 - (\sum x)^2} \times \sqrt{n \sum y^2 - (\sum y)^2}} \]

Table 15.9 Table of calculations (GDP) and unemployment

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>XY</th>
<th>X^2</th>
<th>Y^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>8</td>
<td>120</td>
<td>225</td>
<td>64</td>
</tr>
<tr>
<td>14</td>
<td>9</td>
<td>126</td>
<td>196</td>
<td>81</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td>140</td>
<td>196</td>
<td>100</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
<td>132</td>
<td>144</td>
<td>121</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>132</td>
<td>121</td>
<td>144</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td>96</td>
<td>144</td>
<td>64</td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>105</td>
<td>225</td>
<td>49</td>
</tr>
<tr>
<td>19</td>
<td>5</td>
<td>95</td>
<td>361</td>
<td>25</td>
</tr>
<tr>
<td>22</td>
<td>3</td>
<td>66</td>
<td>484</td>
<td>9</td>
</tr>
<tr>
<td>22</td>
<td>3</td>
<td>66</td>
<td>484</td>
<td>9</td>
</tr>
<tr>
<td>Totals</td>
<td>156</td>
<td>76</td>
<td>1078</td>
<td>2580</td>
</tr>
</tbody>
</table>

\[ n = 10 \]

So \[ r = -0.95 \] (2dp)

c) Coefficient of determination \( R^2 \) is \( r^2 = (-0.95)^2 = 0.90 \)

90% of the variations in unemployment are explained by the variations in GDP. This will form a sound basis for predicting one variable (e.g. unemployment) from the other.

15.3.3 Rank Correlation Coefficient

**DEFINITION**

Rank correlation measures the correlation between variables which have been expressed in form of ranks
There are several ways of calculating the rank correlation coefficient but the most common one is the Spearman’ Rank Correlation Coefficient.

Often denoted by $R$ the Spearman’s rank correlation coefficient is calculated as:

$$R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}$$

where $d =$ difference between corresponding ranks and $n =$ number of pairs of ranks

**Example 9**

The mid semester examination results in Mathematics and Costing of a sample of 6 students were as follow:

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Costing</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>98</td>
<td>77</td>
</tr>
<tr>
<td>Annie</td>
<td>72</td>
<td>84</td>
</tr>
<tr>
<td>Peter</td>
<td>52</td>
<td>50</td>
</tr>
<tr>
<td>Chikondi</td>
<td>65</td>
<td>64</td>
</tr>
<tr>
<td>Mary</td>
<td>45</td>
<td>49</td>
</tr>
<tr>
<td>George</td>
<td>50</td>
<td>20</td>
</tr>
</tbody>
</table>

Required:

Use the Spearman’s rank correlation coefficient to investigate whether or not there is a relationship between ability in Mathematics and Costing.

**Solution**

Since the data given is not in form of ranks, there is need to create ranks and then the rank correlation coefficient formula can be used.

<table>
<thead>
<tr>
<th>Students</th>
<th>Rank in Mathematics</th>
<th>Rank in Costing</th>
<th>$d$</th>
<th>$d^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Annie</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Peter</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Chikondi</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mary</td>
<td>6</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>George</td>
<td>5</td>
<td>6</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

Now $R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}$

$$= 1 - \frac{6 \times 4}{6(6^2 - 1)} = 0.886$$
This is very high correlation which means ability in costing and mathematics are strongly related.

**Tied ranks**

It is possible to have a situation where ranks are tied. For instance there could be 2 number threes of three number fours and so on. Where there are tied ranks, the ties are replaced by the mean of the ranks which would have been there had the ties not occurred.

**Example 10**

A safari operator offers 8 products of adventure to his customers. In order to put more attention to the most popular he has asked two top guides rank the products in terms of excitement generated in customers. The results are as follows:

<table>
<thead>
<tr>
<th>Product</th>
<th>Guide one Ranks</th>
<th>Guide two Ranks</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>F</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>G</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>H</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>

Do the rankings of the two guides corroborate?

**Solution**

Tied ranks for Guide One are: 2 and 2

If no ties these would have been: 2, and 3

Mean (to replace the ties) \[
\frac{2 + 3}{2} = 2.5
\]

Tied ranks for Guide Two: 3, 3 and 3

These took the place of 3, 4 and 5

Mean to replace the ties \[
\frac{3 + 4 + 5}{3} = 4
\]
Table 15.13  Product ranks workings

<table>
<thead>
<tr>
<th>Guide one</th>
<th>Guide two</th>
<th>d</th>
<th>d²</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2.5</td>
<td>1</td>
<td>1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>2.5</td>
<td>4</td>
<td>-1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Shaded ranks are replacements

\[ R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)} \]

\[ = 1 - \frac{6 \times 16.5}{8(8^2 - 1)} = 0.804 \]

The rankings by the two Guides corroborate.

CHAPTER SUMMARY

This chapter established building blocks for investigating relationships between basically two variables, called independent and dependent variables respectively. It studies two related concepts: regression and correlation. Regression analysis searches for a relationship between a variable of interest (dependant variable) and other variables (independent variables). The main goal of such relationship building is to forecast the dependant variable in the future based on past values of the dependent and independent variables. For example, we might want to predict the level of sales in the future for a company by studying the relationship between the sales (dependant variable) and the level of marketing (independent variable).

Correlation, on the other hand, measures the degree or strength of the relationship between the variables. This measure of the strength of the relationship is referred to as the correlation coefficient. Two correlation coefficients were presented: Pearson’s product moment correlation coefficient and Spearman’s rank correlation coefficient. These are calculated using the formula:

Pearson’s product moment correlation coefficient: \[ r = \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{n \sum x^2 - (\sum x)^2}\sqrt{n \sum y^2 - (\sum y)^2}} \]

Spearman’s rank correlation coefficient: \[ R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)} \]

The chapter also looked at the coefficient of determination, \( r^2 \), which is defined as the proportion (or percentage) of variation in the dependent variable, \( y \), that is explained by the independent variable, \( x \).
Once the correlation coefficient, $r$, has been determined, the coefficient of determination is easily found by squaring $r$, i.e. $r^2$.

END OF CHAPTER EXERCISES

1. What is the difference between regression and correlation?

2. Describe any two ways of investigating relationships between variables.

3. In the following set of data, $y$ represents ten finance companies’ total operating costs (in millions of K) for a particular year and $x$ represents the companies’ assets (in millions of K) for the same year.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>310</td>
</tr>
<tr>
<td>3</td>
<td>250</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>450</td>
</tr>
<tr>
<td>2</td>
<td>150</td>
</tr>
<tr>
<td>3</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>320</td>
</tr>
<tr>
<td>3</td>
<td>230</td>
</tr>
<tr>
<td>2</td>
<td>140</td>
</tr>
<tr>
<td>6</td>
<td>400</td>
</tr>
</tbody>
</table>

Required:

a. Draw the scatter diagram on graph paper and comment on the relationship between $x$ and $y$.

b. Find the equation of the least-squares regression line, assuming that operating costs depend on assets.

i. Use your results in (ii) above to predict the operating costs for a firm with assets of K500 million.

ii. If the coefficient of correlation $r$ is 0.93 comment on the likely accuracy of your prediction.

4. A cost accountant has derived the following data on the running costs and distance travelling by twenty of a company’s fleet of new cars used by its computer salesmen last year. Ten of the cars are type F and ten are type L.
<table>
<thead>
<tr>
<th>Distance Travelled (Thousand km)</th>
<th>Running Costs (K million)</th>
<th>Distance Travelled (Thousand km)</th>
<th>Running Costs (K million)</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>y</td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>4.0</td>
<td>5.3</td>
<td>3.5</td>
<td>6.9</td>
</tr>
<tr>
<td>4.6</td>
<td>6.7</td>
<td>4.6</td>
<td>7.6</td>
</tr>
<tr>
<td>5.9</td>
<td>7.5</td>
<td>5.3</td>
<td>7.9</td>
</tr>
<tr>
<td>6.7</td>
<td>8.8</td>
<td>6.0</td>
<td>8.3</td>
</tr>
<tr>
<td>8.0</td>
<td>8.0</td>
<td>7.2</td>
<td>8.8</td>
</tr>
<tr>
<td>8.9</td>
<td>9.1</td>
<td>8.4</td>
<td>9.2</td>
</tr>
<tr>
<td>8.9</td>
<td>10.5</td>
<td>10.1</td>
<td>9.6</td>
</tr>
<tr>
<td>10.1</td>
<td>10.0</td>
<td>11.1</td>
<td>10.3</td>
</tr>
<tr>
<td>10.8</td>
<td>11.7</td>
<td>11.5</td>
<td>10.1</td>
</tr>
<tr>
<td>12.1</td>
<td>12.4</td>
<td>12.3</td>
<td>11.3</td>
</tr>
<tr>
<td>Mean 8.0</td>
<td>9.0</td>
<td>8.0</td>
<td>9.0</td>
</tr>
</tbody>
</table>

**Required:**

(a) The least square regression lines were calculated using a standard computer package as follows:

Car F : \( y = 2.650 + 0.79x \)

Car L : \( y = 5.585 + 0.427x \)

(i) Plot the two scatter diagrams and regression lines on the same graph, distinguishing clearly between the two sets of points.

(ii) Explain the meaning of the two regression coefficients for each set of these data.

(b) For the car F data the following statistics were calculated:

\[
\sum x^2 = 704.34, \quad \sum xy = 771.07, \quad \sum y^2 = 854.38
\]

Calculate the correlation coefficient and interpret its meaning.

(c) Predict the running costs for the two different types of cars if the average distance travelled is 12,000km.

5. Kay’s Confectionaries makes small cakes for functions and an analysis units made in the past and associated costs are as follows.
<table>
<thead>
<tr>
<th>Units Made</th>
<th>Total cost (K'000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>58</td>
</tr>
<tr>
<td>250</td>
<td>145</td>
</tr>
<tr>
<td>90</td>
<td>62</td>
</tr>
<tr>
<td>200</td>
<td>144</td>
</tr>
<tr>
<td>150</td>
<td>85</td>
</tr>
<tr>
<td>50</td>
<td>44</td>
</tr>
<tr>
<td>65</td>
<td>48</td>
</tr>
<tr>
<td>75</td>
<td>50</td>
</tr>
</tbody>
</table>

**Required:**

a) Calculate the Pearson’s product moment correlation coefficient and the coefficient of determination.

b) Interpret your results.

6. (a) Define ‘correlation’.

(b) Calculate Spearman's rank correlation coefficient for the following data and comment on the result:

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>9</td>
<td>6</td>
<td>10</td>
<td>8</td>
</tr>
</tbody>
</table>

7. A Stores supervisor and a Purchase manager were asked to rank the main suppliers (ABCDEFG and H) in order of value to the company. The two managers ranked the eight suppliers and the following are the results.

    Stores supervisor:  E     C     G     H     B     D     A     F
    Purchase manager:  E     G     B     D     C     A     H     F

a) Use the Spearman’s rank correlation coefficient to determine the amount of agreement between the two managers.

b) Can any conclusion be drawn about the suppliers

8. Calculate Spearman’s rank correlation coefficient for the following ordinal data and comment on the result:

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>10</td>
<td>8</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

9. In the following set of data, \( y \) represents ten finance companies’ total operating costs (in millions of K) for a particular year and \( x \) represents the companies’ assets (in millions of K) for the same year.
\[
y \\
5 & 310 \\
3 & 250 \\
2 & 100 \\
5 & 450 \\
2 & 150 \\
3 & 200 \\
4 & 320 \\
3 & 230 \\
2 & 140 \\
6 & 400 \\
35 & 2550
\]

Required:

a) Draw the scatter diagram on graph paper and comment on the relationship between \(x\) and \(y\).

b) If \(\sum xy = 10300\), \(\sum x^2 = 768500\), \(\sum y^2 = 141\)

Find the equation of the least-squares regression line, assuming that operating costs depend on assets.

c) Use your results in (ii) above to predict the operating costs for a firm with assets of £550 million. If \(r = 0.93\) comment on the likely accuracy of your prediction.

Music Technologies, an electronics retail company in Durban, has kept records of the number of Ipods sold within a week of placing advertisements in the Mercury. The following table shows the number of ipods sold and the corresponding number of advertisements placed in the Mercury for 12 randomly selected weeks over the past year.

<table>
<thead>
<tr>
<th>Ads</th>
<th>4</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>5</th>
<th>2</th>
<th>4</th>
<th>3</th>
<th>5</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales</td>
<td>26</td>
<td>28</td>
<td>24</td>
<td>18</td>
<td>35</td>
<td>24</td>
<td>36</td>
<td>25</td>
<td>31</td>
<td>37</td>
<td>30</td>
</tr>
</tbody>
</table>

Required

a) Construct a scatter plot for the data.

b) Find the straight-line regression equation to estimate the number of ipods that Music Centre can expect to sell within a week, based on the number of advertisement placed.

c) Estimate the likely mean sales of ipods when three advertisements are placed.

d) Calculate the product moment correlation coefficient and comment on the result obtained.

e) Calculate the coefficient of determination and interpret the result.
CHAPTER 16  TIME SERIES

LEARNING OBJECTIVES:

By the end of this chapter the student should be able to:

i. Define a ‘time series’
ii. Plot time series data
iii. Describe time series models
iv. Distinguish between components of a time series
v. Decompose a time series into its components
vi. Forecast time series values

16.0 INTRODUCTION

A time series is an **ordered sequence of values of a variable at equally spaced time intervals**. Simply put, time series is a set of data values that are recorded at successive and regular intervals over a period of time e.g. daily, weekly, monthly, quarterly and annually. We must therefore caution you not to think of time series as quarterly data only.

An analysis of history, a time series, can be used by management or business individuals to make current and future plans based on long term time series data, and assuming that patterns would continue into the future. Both short-term and long-term predictions are essential for a business entity to execute possible development plans in terms of financing, procurement, manufacturing, sales, profits, revenue, new products, new plants/machinery, market demand, and recruitment among others.

For forecasts based on time series to be meaningful, the rule of thumb is that the data must be recorded over a relatively longer period of time i.e. at least 10 observations. In this chapter we deal with the use of time series data to forecast future events or activities. We first discuss the components of a time series. Then, we explore basic techniques for analysing time series data in a process called decomposition. Finally, we forecast future activities.

16.1 APPLICATIONS OF TIME SERIES

The analysis of time series has two major objectives namely to

a) Obtain an understanding of the underlying forces and structure that produced the observed data
b) Forecast using fitted time series model. The models can also be used to monitor a set of time series components.

Time Series analysis many areas of application including economic forecasting, sales forecasting, budgetary analysis, stock market analysis, yield projections, process and quality control, inventory studies, workload projections, utility studies, and census analysis.

16.2 PLOTTING A TIME SERIES

A time series is plotted on graph called a **histogram**. The procedure for plotting time series is explained by means of a worked example.
Example 1
The following table shows water consumption (in litres) for a typical household in Kawale.

<table>
<thead>
<tr>
<th>Year</th>
<th>Jan – Mar</th>
<th>Apr – Jun</th>
<th>Jul – Sept</th>
<th>Oct - Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>2009</td>
<td>560</td>
<td>660</td>
<td>740</td>
<td>1200</td>
</tr>
<tr>
<td>2010</td>
<td>600</td>
<td>710</td>
<td>820</td>
<td>1340</td>
</tr>
<tr>
<td>2011</td>
<td>610</td>
<td>750</td>
<td>880</td>
<td>1400</td>
</tr>
<tr>
<td>2012</td>
<td>700</td>
<td>780</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Construct a time series plot for the data

Solution:
To plot time series data, each data value is plotted against the corresponding time point and the points are then joined by straight line segments as shown below.

Figure 16.1 Water consumption

Notice the up and down swings (peaks and troughs) which are typical of time series data. There is marked seasonality in water consumption with the highest water consumption being in the last quarter (Oct-Dec) of each year. Over the years, there is a slight increase in water consumption.

16.3 COMPONENTS OF A TIME SERIES
There are four main components to a time series: the secular trend or simply trend, the cyclical variation, the seasonal variation, and the random variation.

16.3.1 Secular Trend (T):
This is defined as the long-term direction of a time series. The trend can be upward (increasing), downward (decreasing) or constant (show no change) over time. In the previous example, we plotted a time series on a
historigram and noted that there was a slight increase in water consumption over the four-year period. We can confirm this observation by fitting in a trend line in the time series plot as shown below:

Figure 16.2 Trend

16.3.2 Cyclical Variation (C):
The wave-like movement of a time series caused by booms and slumps of an economy over long periods of time. The periods usually correspond to business cycles in the economy.

16.3.3 Seasonal Variation (S):
These are patterns of change in a time series which occur over short repetitive periods. These represent predictable deviation from the trend. For example, an ice creamer seller would expect a substantial increase in sales in summer. Given a particular time point at which data is observed, a seasonal variation is therefore the difference between an actual data value observed and the trend figure.

When we think of season in time series analysis, we must not confine ourselves to annual geographical seasons of say spring, summer, autumn, and winter or rainy and dry seasons. We must note that in business seasons can be monthly (month-end, mid-month), weekly (mid-week, weekend) or even daily (morning, noon, afternoon, evening, night). All these seasons will have varying effects on business operations.

Graphically, we can show the seasonal variation as below:
16.3.4 Random or Irregular Variations (R or I):
These are variations which occur over short intervals and are unpredictable. Factors that can cause these variations include unexpected changes in weather, strikes, breakdowns, theft, war/political unrest, death, and sickness.

16.4 TIME SERIES MODELS
Once the components of a time series are identified, we need to understand how they are combined to provide an observed data value. There are two common models for combining time series components and these are: The **additive model** and the **multiplicative model**.

The additive model assumes that each time series data value is a sum (algebraically) of the components:

\[ Y = T + C + S + R \]

Where
- \( Y \) = Actual observations or observed data value
- \( T \) = Trend
- \( C \) = Cyclical variation
- \( S \) = Seasonal variation
- \( R \) = Random/Irregular/Residual variation

At level of this text we shall ignore Cyclical variations. Random variations will be estimated as a residue in the calculation of average seasonal variations.

Consequently, any time series value shall be assumed to be made up of the remaining two components resulting into:
The multiplicative model also referred to as the classical time series model assumes that each time series data value is as a result of multiplying (a product of) the components:

\[ Y = T \times C \times S \times R \]

If we ignore the C and R components, the model becomes

\[ Y = T \times S \]

**Note:** At Technician Level candidates are normally told which model to use. However, in the event that the model is not specified, full marks are awarded if the time series is analysed correctly by using either model. Mostly, candidates are expected to use the additive model. At Foundation Level, however, candidates are expected to be able to decide from a plot of the data or from other methods which model to apply, and then use the correct model.

### 16.5 TIME SERIES DECOMPOSITION

Decomposition is the process of breaking a time series into its components. As pointed out earlier, our focus in this manual is on the trend and seasonal components.

#### 16.5.1 The Trend

The objectives behind the study of the trend are:

- To explain the general underlying tendency in the movement of data over time. This is usually assumed to be in form of a linear pattern with a positive (increasing) or negative (decreasing) gradients.
- To remove the trend from the data in order to expose the movements in the other components.

The trend can be estimated using any of the following methods:

a) Semi-average method  
b) Least squares method  
c) Moving average method

**A. Trend by Semi-Average Method**

To find the trend using the semi-average method we follow the following basic steps:

**Step 1:**
Split the data into 2 equal parts (halves). If there is an odd number of observations/values, simply leave out the median values.

**Step 2:**
Calculate the arithmetic mean for each group to obtain \( \bar{x}_1 \) and \( \bar{x}_2 \).

**Step 3:**
Plot these two points against the median position of each group. Join the two points with a straight line. This is the trend line.

The trend for each time point is then read from the linear plot. Alternatively, find the average increase/decrease per year by dividing the difference between the two mean points by the number of time points between them. We illustrate this by an example.
Example 2
The following data shows the quarterly domestic exports (to the nearest billion of kwacha) for Malawi for the years 2010 to 2012.

Table 16.2 Domestic Exports

<table>
<thead>
<tr>
<th>Year</th>
<th>Domestic exports 2010 – 2012</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Jan-Mar</td>
</tr>
<tr>
<td>2010</td>
<td>32</td>
</tr>
<tr>
<td>2011</td>
<td>48</td>
</tr>
<tr>
<td>2012</td>
<td>50</td>
</tr>
</tbody>
</table>

Source: 2011 & 2012 Malawi Trade Briefs - NSO

Find the trend by means of semi-average method. And hence assign trend values for all quarters.

Solution

Step 1: Splitting the data
The data set has 11 observations. Therefore it can be split into halves of 5 observations each. The first half covers the period Jan-Mar 2010 to Jan-Mar 2011 with the following export values: 32, 27, 52, 49 and 48. The second half covers the period Jul-Sep 2011 to Jul-Sep 2012 with the following export values: 53, 76, 50, 65 and 100.

Note: Since have an odd number of values, we have left out the median data observation of 46 for Apr-Jun 2011.

Step 2: Calculating the arithmetic means
\[
\bar{x}_1 = \frac{32 + 27 + 52 + 49 + 48}{5} = 41.6, \quad \bar{x}_2 = \frac{53 + 76 + 50 + 65 + 100}{5} = 68.8
\]

Step 3: Plotting the mean points
We will plot the time series data just as before and then fit in a trend line by joining the two mean points. \(\bar{x}_1\) will be plotted against Jul-Sep 2010 and \(\bar{x}_2\) against Jan-Mar 2012. These are the median positions for each group.
Figure 16.4   Domestic Exports – Trend by semi averages

The trend values for each time point can be read from the graph or calculated as below:

Average change in trend = \( \frac{\text{Difference between the mean points}}{\text{Number time points between them}} \)

\[ \frac{\bar{x}_2 - \bar{x}_1}{6} = \frac{68.6 - 41.6}{6} = 4.53 \]

We will proceed to find the trend values by adding or subtracting 4.53 to or from either of the mean points. For example the trend for Apr-Jun 2010 is 41.6 – 4.53 = 37.1, for Oct-Dec 2010 is 41.6 + 4.53 = 43.1, and for Apr-June 2012 is 68.8 + 4.53 = 73.3. The rest of the trend values are presented below.

<table>
<thead>
<tr>
<th>Year</th>
<th>Month</th>
<th>Trend</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>Jan-Mar</td>
<td>32.5</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>37.1</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>41.6</td>
</tr>
<tr>
<td></td>
<td>Oct-Dec</td>
<td>46.1</td>
</tr>
<tr>
<td>2011</td>
<td>Jan-Mar</td>
<td>50.7</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>55.2</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>59.7</td>
</tr>
<tr>
<td></td>
<td>Oct-Dec</td>
<td>64.3</td>
</tr>
<tr>
<td>2012</td>
<td>Jan-Mar</td>
<td>68.8</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>73.3</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>77.9</td>
</tr>
<tr>
<td></td>
<td>Oct-Dec</td>
<td>-</td>
</tr>
</tbody>
</table>
B. Trend by Least Squares Method
To find the trend using the method of least squares we follow the following steps:

Step 1:
Treat time points as the independent X-variable. The time points are transformed into values through a process of coding as follows: The first time point is coded 1, the second time point is coded 2, and so on until the last time point is coded.

Step 2:
Obtain a least squares regression line between the x-codes and the time series data values (Y-values). The trend line through the data is given by:

\[ y = a + bx \]

Where:

\[ y = \text{trend value for a given period} \]
\[ b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2} \]
\[ a = \frac{\sum y}{n} - b \frac{\sum x}{n} \]

To calculate the trend value, substitute the appropriate x-code into the least squares equation and compute the value of y.

Example 3
Find the trend by means of least squares method for the domestic export data from the previous example.

Solution
To find the least squares trend, first code the time point as follows: Jan-Mar 2010 – 1, Apr-Jun 2010 – 2, Jul-Sep 2010 – 3, … Jul-Sep 2012 – 11.

Table 16.4 Trend calculation (least squares)

| Year | Month   | X-codes | Data value Y | Trend  \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>Jan-Mar</td>
<td>1</td>
<td>32</td>
<td>29.6</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>2</td>
<td>27</td>
<td>34.5</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>3</td>
<td>52</td>
<td>39.5</td>
</tr>
<tr>
<td></td>
<td>Oct-Dec</td>
<td>4</td>
<td>49</td>
<td>44.4</td>
</tr>
<tr>
<td>2011</td>
<td>Jan-Mar</td>
<td>5</td>
<td>48</td>
<td>49.4</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>6</td>
<td>46</td>
<td>54.3</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>7</td>
<td>53</td>
<td>59.3</td>
</tr>
<tr>
<td></td>
<td>Oct-Dec</td>
<td>8</td>
<td>76</td>
<td>64.2</td>
</tr>
<tr>
<td>2012</td>
<td>Jan-Mar</td>
<td>9</td>
<td>50</td>
<td>69.2</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>10</td>
<td>65</td>
<td>74.1</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>11</td>
<td>100</td>
<td>79.1</td>
</tr>
</tbody>
</table>
To find the least squares line, \( y = a + bx \), we need the following:

\[
\begin{align*}
    n &= 11, \quad \sum x = 66, \quad \sum x^2 = 506, \quad \sum y = 598, \quad \sum xy = 4133 \\
    \therefore b &= \frac{11 \times 4133 - 66 \times 598}{11 \times 506 - 66^2} = 4.9545 \quad \text{and} \quad a = \frac{598}{11} - 4.9545 \times \frac{66}{11} = 24.6366
\end{align*}
\]

The least square trend is therefore given by \( y = 24.64 + 4.95x \).

We calculate the trend values by substituting the x-codes into the least squares equation, \( y = 24.64 + 4.95x \). For example, for Jan-Mar 2011 (coded 5) the trend is \( y = 24.64 + 4.95 \times 5 \approx 49.4 \). The rest of the trend values are provided in the last column of the table above.

**Trend by Moving Averages Method**

**C. Trend by Moving Averages Method**

This is an alternative and possibly standard method for finding the trend and requires no specific mathematical formula. The method is non-linear (though fairly linear), in the sense that it does not result in a straight line, but it does smooth out peaks and valleys (ups and downs) in a set of observations by “removing” seasonal and random variations from to reveal the trend.

The moving average trend is accomplished by “moving” the arithmetic mean values through the time series i.e. averaging sets of overlapping data observations through the time series data. In working out the moving average trend, the number of observations to include or average depends on the periodicity of the time series. With quarterly data, the period is 4; for data recorded daily 5 days a week, the period would be 5. If the period is an **even** number (e.g. quarterly data or data recorded daily for 6 days/week) then a **centred moving average** is required. However, if the period is an **odd** number (e.g. data recorded daily 5 or 7 days/week), then a **simple moving average** is appropriate.

**Calculation of a moving average**

As indicated earlier, a moving average is calculated by averaging a set of “consecutive” and overlapping values at a time. We illustrate the procedure in the following example.

**Example 4**

The following are production figures over 9 days. Use them to find 2-point and 3-point moving averages.

<table>
<thead>
<tr>
<th>Day</th>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
<th>Day 6</th>
<th>Day 7</th>
<th>Day 8</th>
<th>Day 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production</td>
<td>6</td>
<td>4</td>
<td>10</td>
<td>8</td>
<td>5</td>
<td>12</td>
<td>9</td>
<td>7</td>
<td>15</td>
</tr>
</tbody>
</table>

**Solution**

For 2-point moving averages:
We first find the moving totals by adding 2 adjacent figures at a time and allow the pairs to overlap through the last figure. The first moving total is $6 + 4 = 10$, Next we drop the first figure 6, and then add 3rd figure 10: $4 + 10 = 14$, and so on. These moving totals must be placed at the median position as shown in the 3rd column of the following table. To obtain the 2-point moving averages, we divide the moving totals by 2. The moving averages are placed in column 4.

Table 16.6  Production figures – moving averages

<table>
<thead>
<tr>
<th>Day</th>
<th>Data value, y</th>
<th>Moving totals</th>
<th>2-point moving average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td>6</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Day 2</td>
<td>4</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>Day 3</td>
<td>10</td>
<td>18</td>
<td>9</td>
</tr>
<tr>
<td>Day 4</td>
<td>8</td>
<td>13</td>
<td>6.5</td>
</tr>
<tr>
<td>Day 5</td>
<td>5</td>
<td>17</td>
<td>8.5</td>
</tr>
<tr>
<td>Day 6</td>
<td>12</td>
<td>21</td>
<td>10.5</td>
</tr>
<tr>
<td>Day 7</td>
<td>9</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>Day 8</td>
<td>7</td>
<td>22</td>
<td>11</td>
</tr>
<tr>
<td>Day 9</td>
<td>15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For 3-point moving averages:

We first find the moving totals by adding 3 adjacent figures at a time and allow the sets to overlap through the last figure. The first moving total is $6 + 4 + 10 = 20$, Next we drop the first figure 6, and then add 4th figure 8: $4 + 10 + 8 = 22$, and so on. These moving totals must be placed at the median position as shown in the 3rd column of the following table. To obtain the 3-point moving averages, we divide the moving totals by 3. The moving averages are placed in column 4.

Table 16.7  Production figures (moving averages)

<table>
<thead>
<tr>
<th>Day</th>
<th>Data value, y</th>
<th>Moving totals</th>
<th>2-point moving average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td>6</td>
<td>10</td>
<td>6.67</td>
</tr>
<tr>
<td>Day 2</td>
<td>4</td>
<td>20</td>
<td>7.33</td>
</tr>
<tr>
<td>Day 3</td>
<td>10</td>
<td>22</td>
<td>7.67</td>
</tr>
<tr>
<td>Day 4</td>
<td>8</td>
<td>23</td>
<td>8.33</td>
</tr>
<tr>
<td>Day 5</td>
<td>5</td>
<td>25</td>
<td>8.67</td>
</tr>
<tr>
<td>Day 6</td>
<td>12</td>
<td>26</td>
<td>9.33</td>
</tr>
<tr>
<td>Day 7</td>
<td>9</td>
<td>28</td>
<td>10.33</td>
</tr>
<tr>
<td>Day 8</td>
<td>7</td>
<td>31</td>
<td></td>
</tr>
<tr>
<td>Day 9</td>
<td>15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Note:** In a 2-point moving average the moving totals and averages fall between two time points (days). For example the first is between the Day 1 and Day 2 being the median position of the first pair of figures. The 3-point moving totals and averages fall against/adjacent times points. For example the first is aligned against Day 2 being the median position for the first three figures. Notice that the resulting moving averages are less variable than the original data. The production figures have been ‘smoothed’.

The most common moving averages calculated in time series data for commercial purposes are 4, 5, 6 or 7 point ones because most data relates to 4 quarter/yearly cycles, and 5, 6 or 7 day/weekly cycles.

**Example 5 (Data with an even period)**
The following data shows the sales revenue (in MK’ million) of a local company from 2010 to 2013.

<table>
<thead>
<tr>
<th>Year</th>
<th>Qtr 1</th>
<th>Qtr 2</th>
<th>Qtr 3</th>
<th>Qtr 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>79</td>
<td>48</td>
<td>68</td>
<td>107</td>
</tr>
<tr>
<td>2011</td>
<td>97</td>
<td>66</td>
<td>85</td>
<td>134</td>
</tr>
<tr>
<td>2012</td>
<td>113</td>
<td>91</td>
<td>100</td>
<td>148</td>
</tr>
<tr>
<td>2013</td>
<td>136</td>
<td>105</td>
<td>125</td>
<td>174</td>
</tr>
</tbody>
</table>

Find the trend using the moving average method and plot it together with the time series data.

**Solution**
We proceed to arrange the data in a table that will help facilitate the calculation of trend values. Place the data values in one column (in our case column 2). Then add in fours since the period is 4 (for quarterly data): The first moving total is 79+48+68+107=302, and for the second we drop 79 and then add the fifth value 97: 48+68+107+97=320 and so on. The rest of the moving totals are place in column 2 in the table. The moving totals are then divided by 4 (the period) to obtain the moving averages. These are placed in column 4. Notice that both the moving totals and averages are not aligned with time points (Qtrs) because the period is even. Hence the need to find centred moving averages for the ‘trend’. To find the centred moving averages we add the moving averages in pairs and divide by 2. The first centred moving average is (79+80)÷2=79.5≈79.8, the second is (80+84.5)÷2=82.25≈82.3, and so on. Each centred moving averages is placed between the two moving averages being averaged. This then aligns the centred moving averages (Trend) with the time points (Qtrs) as shown in column 5 in the following table.
### Table 16.9  
Sales revenue Trend by moving averages

<table>
<thead>
<tr>
<th>Year/Qtr</th>
<th>Data value</th>
<th>Moving totals (add in fours)</th>
<th>Moving average (Total divide by 4)</th>
<th>‘Trend’ Centred average (add in pairs and divided by 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010 Qtr 1</td>
<td>79</td>
<td>302</td>
<td>75.50</td>
<td>77.8</td>
</tr>
<tr>
<td>Qtr 2</td>
<td>48</td>
<td></td>
<td></td>
<td>80.00</td>
</tr>
<tr>
<td>Qtr 3</td>
<td>68</td>
<td>320</td>
<td>84.50</td>
<td>82.3</td>
</tr>
<tr>
<td>Qtr 4</td>
<td>107</td>
<td>338</td>
<td>88.75</td>
<td>86.6</td>
</tr>
<tr>
<td>2011 Qtr 1</td>
<td>97</td>
<td>355</td>
<td>92.1</td>
<td>102.6</td>
</tr>
<tr>
<td>Qtr 2</td>
<td>66</td>
<td>382</td>
<td>95.50</td>
<td>97.5</td>
</tr>
<tr>
<td>Qtr 3</td>
<td>85</td>
<td>398</td>
<td>100.0</td>
<td>105.75</td>
</tr>
<tr>
<td>Qtr 4</td>
<td>134</td>
<td>423</td>
<td>109.50</td>
<td>107.6</td>
</tr>
<tr>
<td>2012 Qtr 1</td>
<td>113</td>
<td>438</td>
<td>111.3</td>
<td>113.00</td>
</tr>
<tr>
<td>Qtr 2</td>
<td>91</td>
<td>452</td>
<td>115.9</td>
<td>118.75</td>
</tr>
<tr>
<td>Qtr 3</td>
<td>100</td>
<td>475</td>
<td>120.5</td>
<td>122.25</td>
</tr>
<tr>
<td>Qtr 4</td>
<td>148</td>
<td>489</td>
<td>125.4</td>
<td></td>
</tr>
<tr>
<td>2013 Qtr 1</td>
<td>136</td>
<td>514</td>
<td>128.50</td>
<td>131.8</td>
</tr>
<tr>
<td>Qtr 2</td>
<td>105</td>
<td>540</td>
<td>135.00</td>
<td></td>
</tr>
<tr>
<td>Qtr 3</td>
<td>105</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Qtr 4</td>
<td>174</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Both the time series data values and the trend values are plotted on the following graph.
Notice that the moving average trend has smoothed out the time series and is fairly linear.

**Example 7 (Data with an odd period)**
The following data shows the output of a factory located in Zolozolo over a 3-week period.

<table>
<thead>
<tr>
<th>Week</th>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td>86</td>
<td>92</td>
<td>104</td>
<td>98</td>
<td>78</td>
</tr>
<tr>
<td>Week 2</td>
<td>90</td>
<td>99</td>
<td>110</td>
<td>102</td>
<td>80</td>
</tr>
<tr>
<td>Week 3</td>
<td>93</td>
<td>101</td>
<td>116</td>
<td>110</td>
<td>83</td>
</tr>
</tbody>
</table>

By means of moving average method, find the trend.

**Solution:**
We proceed as in the previous example. Note that the period is now 5 (an odd number); hence we do not need to centre the moving averages. For instance the first moving total of 485 (i.e. $86+92+104+98+78$) is placed at the median position of the set which is alongside 104 i.e. Wednesday of Week 1. The second moving total 462 (i.e. $92+104+98+78+90$) is placed alongside 98 i.e. Thursday of Week 2, and so on. The moving totals and simple moving averages are provided in the table that follows. The simple moving averages are our trend values.
Table 16.11 Zolozolo factory output trend

<table>
<thead>
<tr>
<th>Week/Day</th>
<th>Data value</th>
<th>Moving totals (add in fives)</th>
<th>‘Trend’ Moving average (Total divide by 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>86</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tue</td>
<td>92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wed</td>
<td>104</td>
<td>458</td>
<td>91.6</td>
</tr>
<tr>
<td>Thu</td>
<td>98</td>
<td>462</td>
<td>92.4</td>
</tr>
<tr>
<td>Fri</td>
<td>78</td>
<td>469</td>
<td>93.8</td>
</tr>
<tr>
<td>Week 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>90</td>
<td>475</td>
<td>95.0</td>
</tr>
<tr>
<td>Tue</td>
<td>99</td>
<td>479</td>
<td>95.8</td>
</tr>
<tr>
<td>Wed</td>
<td>110</td>
<td>481</td>
<td>96.2</td>
</tr>
<tr>
<td>Thu</td>
<td>102</td>
<td>484</td>
<td>96.8</td>
</tr>
<tr>
<td>Fri</td>
<td>80</td>
<td>486</td>
<td>97.2</td>
</tr>
<tr>
<td>Week 3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>93</td>
<td>492</td>
<td>98.4</td>
</tr>
<tr>
<td>Tue</td>
<td>101</td>
<td>500</td>
<td>100.0</td>
</tr>
<tr>
<td>Wed</td>
<td>116</td>
<td>503</td>
<td>100.6</td>
</tr>
<tr>
<td>Thu</td>
<td>110</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fri</td>
<td>83</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

16.5.2 Seasonal Variation

As discussed earlier, many time series are affected by seasonal factors. It is necessary to find the value of a seasonal component in order to obtain seasonally adjusted data and for forecasting purposes. The procedure for finding the seasonal variations/factors depends on the model used.

Using the additive model:
- Obtain the seasonal variation for each time point by subtracting the ‘trend’ from the observed data value i.e. \( S = Y - T \)
- Find the average seasonal variation for time point. Check that the sum of the average seasonal variations is equal to zero (0).
- If the sum of the averages is not equal to zero, then the averages must be adjusted accordingly so that they sum to zero. The adjustment factor is \( \frac{\text{Sum of the averages}}{\text{Period}} \). If the sum is less than zero, we add the modulus of the adjustment factor to each average. However, if the sum is greater than zero, we subtract the adjustment factor from each average to obtain the adjusted seasonal variations/factors.

Using the multiplicative model:
- Obtain the seasonal variation for each time point by dividing the data value by the ‘trend’ i.e. \( S = \frac{Y}{T} \)
- Find the average seasonal variation for each time point. Check that the sum of the average seasonal variations is equal to period. For quarterly data, they must sum to 4, for a 5 day-week, they must sum to 5.

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If the sum of the averages does not equal the period, then the averages must be adjusted accordingly so that their sum equals the period. The adjustment factor is \( \frac{\text{Period}}{\text{Sum of the averages}} \). The adjusted seasonal variations/factors are then obtained by multiplying each average by the adjustment factor.

**Determining which model to use**

The additive model is used when the seasonal variation is approximately the same, irrespective of the trend values. Consider the graph of sales revenue for a local company which we presented before and is reproduced here.

Figure 16.6  
Sales revenue of a local company  2011-2013

We can see that the seasonal variations (differences between observed and trend values) for similar quarters are almost the same. In this case the additive model is appropriate.

In the multiplicative model the season variation is approximately proportional to the trend. If the trend is upward, then the variation increases. If the trend is downward, then the seasonal variation decreases. Consider the following graph:
can see that the trend is upward and the seasonal variations (differences between observed and trend values) for similar quarters are increasing (in absolute terms) as we move from 2010 to 2012. In this case multiplicative model is used.

Apart from the graphical method, the other and quicker method for decide the right model to use is take the difference between the smallest and largest value for each year (or week). If the differences are roughly the same or do not follow a particular pattern, then additive model is appropriate. However, if the differences increasing for an upward trend or decreasing for a downward trend, then the multiplicative model is appropriate. Consider the data sets whose graphs we just compared.

Table 16.12 Sale revenue figures

<table>
<thead>
<tr>
<th>Year</th>
<th>Qtr 1</th>
<th>Qtr 2</th>
<th>Qtr 3</th>
<th>Qtr 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>79</td>
<td>48</td>
<td>68</td>
<td>107</td>
</tr>
<tr>
<td>2011</td>
<td>97</td>
<td>66</td>
<td>85</td>
<td>134</td>
</tr>
<tr>
<td>2012</td>
<td>113</td>
<td>91</td>
<td>100</td>
<td>148</td>
</tr>
<tr>
<td>2013</td>
<td>136</td>
<td>105</td>
<td>125</td>
<td>174</td>
</tr>
</tbody>
</table>

differences are: 2010: 107 – 48 = 59  
2011: 134 – 66 = 68  
2012: 148 – 91 = 57  
2013: 174 – 105 = 69

While the trend is upward, the differences show no obvious pattern. Therefore additive model is appropriate.
Table 16.13  Number of visitors 2010 - 2012

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of visitors 2010-2012</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Qtr 1</td>
</tr>
<tr>
<td>2010</td>
<td>24</td>
</tr>
<tr>
<td>2011</td>
<td>32</td>
</tr>
<tr>
<td>2012</td>
<td>36</td>
</tr>
</tbody>
</table>

The differences are:  
2010: 48 – 24 = 24  
2011: 66 – 32 = 34  
2012: 96 – 36 = 58

As the trend is upward and the differences are increasing, the multiplicative model should be appropriate.

Note: In an examination the applicable model will be specified for TC3 paper.

Example 8 (Continuation of Example 5)
Using the sales revenue data and the trend calculated in the previous example find the quarterly seasonal variation using the additive model.

Solution:
To find seasonal variations we need both the observations and trend values from the previous example. We reproduce the table from the previous example here:

Table 16.14  Calculation table - Sales revenue seasonal variations

| Year/Qtr | Data value | Moving totals | Moving average | ‘Trend’ Centred average | Seasonal variation $S = Y − T$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2010 Qtr 1</td>
<td>79</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Qtr 2</td>
<td>48</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Qtr 3</td>
<td>68</td>
<td>77.8</td>
<td>68 – 77.8 = -9.8</td>
<td></td>
</tr>
<tr>
<td>Qtr 4</td>
<td>107</td>
<td>82.3</td>
<td>107 – 82.3 = 24.7</td>
<td></td>
</tr>
<tr>
<td>2011 Qtr 1</td>
<td>97</td>
<td>86.6</td>
<td>10.4</td>
<td></td>
</tr>
<tr>
<td>Qtr 2</td>
<td>66</td>
<td>92.1</td>
<td>-26.1</td>
<td></td>
</tr>
<tr>
<td>Qtr 3</td>
<td>85</td>
<td>97.5</td>
<td>-12.5</td>
<td></td>
</tr>
<tr>
<td>Qtr 4</td>
<td>134</td>
<td>102.6</td>
<td>31.4</td>
<td></td>
</tr>
<tr>
<td>2012 Qtr 1</td>
<td>113</td>
<td>107.6</td>
<td>5.4</td>
<td></td>
</tr>
<tr>
<td>Qtr 2</td>
<td>91</td>
<td>111.3</td>
<td>-20.3</td>
<td></td>
</tr>
<tr>
<td>Qtr 3</td>
<td>100</td>
<td>115.9</td>
<td>-15.9</td>
<td></td>
</tr>
<tr>
<td>Qtr 4</td>
<td>148</td>
<td>120.5</td>
<td>27.5</td>
<td></td>
</tr>
<tr>
<td>2013 Qtr 1</td>
<td>136</td>
<td>125.4</td>
<td>10.6</td>
<td></td>
</tr>
<tr>
<td>Qtr 2</td>
<td>105</td>
<td>131.8</td>
<td>-26.8</td>
<td></td>
</tr>
<tr>
<td>Qtr 3</td>
<td>125</td>
<td></td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Qtr 4</td>
<td>174</td>
<td></td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Notice that we have left out the moving totals and moving averages. This is so because we do not need them at this stage.
Having subtracted the trend values (T) from the time series values (Y), we must arrange the differences (i.e. seasonal variations) such that all seasonal variations for similar quarters are in the same column.

Table 16.15 calculation of average seasonal variations

<table>
<thead>
<tr>
<th>Year</th>
<th>Seasonal variation (SV)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Qtr 1</td>
</tr>
<tr>
<td>2010</td>
<td>-</td>
</tr>
<tr>
<td>2011</td>
<td>10.4</td>
</tr>
<tr>
<td>2012</td>
<td>5.4</td>
</tr>
<tr>
<td>2013</td>
<td>10.6</td>
</tr>
<tr>
<td>Total</td>
<td>26.4</td>
</tr>
<tr>
<td>Average SV</td>
<td>8.8</td>
</tr>
</tbody>
</table>

Adjustment factor

\[
\text{Sum} \quad \quad \quad \text{Period} = \frac{-0.4}{4} = -0.1
\]

Adjusted SV

8.9  -24.3  -12.6  28.0

Seasonal variation are in MK million

The final seasonal variations are:

<table>
<thead>
<tr>
<th>Quarter 1</th>
<th>Quarter 2</th>
<th>Quarter 3</th>
<th>Quarter 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>MK9m</td>
<td>MK24m</td>
<td>MK13</td>
<td>MK28m</td>
</tr>
</tbody>
</table>

Note:
- To obtain the average, divide the totals by the number of seasonal variations in the respective column. In the case above, the totals have been divided by 3 since each column has three seasonal variations. In some cases the numbers will differ from column to column.
- The adjusted seasonal variations must be given be given to the same accuracy as the original data.

Seasonally Adjusted or Deseasonalized data
Most economic and business time series are published seasonally adjusted i.e. with seasonal variations removed.
The additive model

For additive model, seasonally adjusted values are obtained by subtracting the adjusted seasonal variations from respective time series values (Actual)

\[ Y_{\text{Seasonally adjusted}} = Y - S_{\text{Adjusted}} \]

For example, for 1st Quarter 2010, the seasonally adjusted value is 79 – 9 = 70 i.e. K70 million, and for 2nd Quarter, 2010 is 48 – (-24) = 72 i.e. K72 million.

The Multiplicative model

In the case of the multiplicative model, seasonally adjusted values are obtained by dividing the time series values by the respective adjusted seasonal variations:

\[ Y_{\text{Seasonally adjusted}} = \frac{Y}{S_{\text{Adjusted}}} \]

Example 9 (Continuation of Example 5)

Using the sales revenue data, the trend and seasonal variations calculated in the previous examples, seasonally adjust the sales revenue for 2013 (using the additive model).

Solution:

We need to subtract the average seasonal variations from the 2013 sales revenues

<table>
<thead>
<tr>
<th>2013</th>
<th>Data, Y</th>
<th>Seasonal variation, S</th>
<th>Seasonally adjusted sales</th>
</tr>
</thead>
<tbody>
<tr>
<td>Qtr 1</td>
<td>136</td>
<td>9</td>
<td>136 – 9 = 127</td>
</tr>
<tr>
<td>Qtr 2</td>
<td>105</td>
<td>-24</td>
<td>105 – (-24) = 129</td>
</tr>
<tr>
<td>Qtr 3</td>
<td>125</td>
<td>-13</td>
<td>125 – (-13) = 138</td>
</tr>
<tr>
<td>Qtr 4</td>
<td>174</td>
<td>28</td>
<td>174 – 28 = 146</td>
</tr>
</tbody>
</table>

The seasonally adjusted sales for 2013 are: 1st Quarter – K127 million, 2nd Quarter – K129 million, 3rd Quarter – K138 million and 4th Quarter – K146 million

Interpretation:

These are the ‘sales that the company would have made in each quarter of 2013 if it were not for the effects of seasons.
seasons tend to bring the sales down.

### 16.6 FORECASTING

To obtain forecasts, it is necessary to project/extrapolate the trend. Projected/extrapolated trend values are obtained based on the method employed to find the trend.

Using least squares method:
- Extend the x-coding to the period whose forecast is sought.
- Substitute the appropriate x-code in the least squares equation to obtain the trend estimate.

Using moving average method:
- Find the average trend increase (or decrease) i.e.
  \[
  \text{Average increase in trend} = \frac{T_n - T_1}{n-1}
  \]
- Add the average trend increase to the last trend values until the required trend estimates are obtained.

To forecast future values, either add the projected trends to the respective adjusted seasonal variations (for additive model) or multiply the projected trends with the respective seasonal variations (for multiplicative model).

**Example 10 (Continuation of Example 9)**

Using the sales revenue data, the trend and seasonal variations calculated in the previous examples, forecast the sales revenue for each quarter of 2014 (using the additive model).

**Solution:**

We first find the average quarterly increase in trend. The average trend increase is \((\text{Last trend}, 131.8 - \text{first trend}, 77.8)\) divided by number of increments \((12 - 1)\) where 12 is the number of trend.

\[
\text{Average increase in trend} = \frac{131.8 - 77.8}{12 - 1} = \frac{54}{11} = 4.9
\]

Since the last trend, 131.8, corresponds to the Qtr 2, 2013, to find the trend estimate for the first quarter, Qtr 1, of 2014, we need to add 4.9 three times to the 131.8. The projected trend value is given by

\[
131.8 + 3 \times 4.9 = 146.5
\]

For the remaining quarters of 2014, the projected trend values are:
- Qtr 2 trend: 131.8 + 4 \times 4.9 = 151.4
- Qtr 3 trend: 131.8 + 5 \times 4.9 = 156.3
- Qtr 4 trend: 131.8 + 6 \times 4.9 = 161.2

We proceed to obtain the forecast for each quarter by adding the adjusted seasonal variations as discussed earlier.
Table 16.18  Forecasting

<table>
<thead>
<tr>
<th>2014</th>
<th>Trend, $T_{Estimated}$</th>
<th>Seasonal variation, $S$</th>
<th>Forecast = $T + S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Qtr 1</td>
<td>146.5</td>
<td>9</td>
<td>146.5 + 9 = 155.5</td>
</tr>
<tr>
<td>Qtr 2</td>
<td>151.4</td>
<td>-24</td>
<td>151.4 + (-24) = 127.4</td>
</tr>
<tr>
<td>Qtr 3</td>
<td>156.3</td>
<td>-13</td>
<td>156.3 + (-13) = 143.3</td>
</tr>
<tr>
<td>Qtr 4</td>
<td>161.2</td>
<td>28</td>
<td>161.2 + 28 = 189.2</td>
</tr>
</tbody>
</table>

Our forecasts for 2014 are: 1st Quarter – K156 million, 2nd Quarter – K127 million, 3rd Quarter – K143 million, and 4th Quarter – K189 million.

**Note**
The forecasts assume that there is a linear trend, and that the projection of a linear trend reflects the future. The seasonal variations are also assumed to be stable. However, if these assumptions are not true, then our forecasts are not correct.

This far we have successfully managed to look at problems involving quarterly data and the additive model. We now shift our attention to time series data other quarterly data and the multiplicative model.

**Example 11:**
The owner of Zathu, a popular restaurant in Lilongwe wishes to study the patterns of customer numbers for the days of the week in order to be able to forecast activities for the coming days. He then gathers data on patronage over three weeks as follows.

Table 16.19  Customers to Zathu

<table>
<thead>
<tr>
<th>Week</th>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td>113</td>
<td>72</td>
<td>65</td>
<td>95</td>
<td>145</td>
</tr>
<tr>
<td>Week 2</td>
<td>118</td>
<td>81</td>
<td>77</td>
<td>110</td>
<td>160</td>
</tr>
<tr>
<td>Week 3</td>
<td>129</td>
<td>100</td>
<td>95</td>
<td>120</td>
<td>166</td>
</tr>
</tbody>
</table>

a) Calculate the trend by using the method of moving averages.
b) Calculate the adjusted seasonal variations assuming the additive model.
c) Forecast the number customer for Monday and Tuesday of Week 4.
d) Calculate the adjusted seasonal variations assuming the multiplicative model.
e) Forecast the number of customers for Monday and Tuesday of Week 4.
Solution:

a) Trend using moving average:

Since the time series is of period 5 and 5 is an odd number, simple 5-point moving averages are required for the trend. We will add the data values in fives, place the total alongside the median position and then divide each moving total by 5 as shown below.

Table 16.20 Calculation table - Trend and SV

<table>
<thead>
<tr>
<th>Week/Day</th>
<th>Data value</th>
<th>Moving totals (add in fives)</th>
<th>‘Trend’ Moving average (Total divide by 5)</th>
<th>SV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1 Mon</td>
<td>113</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Tue</td>
<td>72</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Wed</td>
<td>65</td>
<td>490</td>
<td>98</td>
<td>-33</td>
</tr>
<tr>
<td>Thu</td>
<td>95</td>
<td>495</td>
<td>99</td>
<td>-4</td>
</tr>
<tr>
<td>Fri</td>
<td>145</td>
<td>504</td>
<td>100.8</td>
<td>44.2</td>
</tr>
<tr>
<td>Week 2 Mon</td>
<td>118</td>
<td>516</td>
<td>103.2</td>
<td>14.8</td>
</tr>
<tr>
<td>Tue</td>
<td>81</td>
<td>531</td>
<td>106.2</td>
<td>-25.2</td>
</tr>
<tr>
<td>Wed</td>
<td>77</td>
<td>546</td>
<td>109.2</td>
<td>-32.2</td>
</tr>
<tr>
<td>Thu</td>
<td>110</td>
<td>557</td>
<td>111.4</td>
<td>-1.4</td>
</tr>
<tr>
<td>Fri</td>
<td>160</td>
<td>576</td>
<td>115.2</td>
<td>44.8</td>
</tr>
<tr>
<td>Week 3 Mon</td>
<td>129</td>
<td>594</td>
<td>118.8</td>
<td>10.2</td>
</tr>
<tr>
<td>Tue</td>
<td>100</td>
<td>604</td>
<td>120.8</td>
<td>-20.8</td>
</tr>
<tr>
<td>Wed</td>
<td>95</td>
<td>610</td>
<td>122</td>
<td>-27</td>
</tr>
<tr>
<td>Thu</td>
<td>120</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Fri</td>
<td>166</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

b) Seasonal variations, SV using additive model
Seasonal variation, $S = Y - T$

Table 16.21 Seasonal variations using additive model

<table>
<thead>
<tr>
<th>Week</th>
<th>Seasonal variation (SV)</th>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td>-</td>
<td>-72</td>
<td>-33</td>
<td>-4</td>
<td>44.2</td>
<td></td>
</tr>
<tr>
<td>Week 2</td>
<td>14.8</td>
<td>25</td>
<td>-25.2</td>
<td>-32.2</td>
<td>-1.4</td>
<td>44.8</td>
</tr>
<tr>
<td>Week 3</td>
<td>10.2</td>
<td>27</td>
<td>-27</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>25</td>
<td>-46</td>
<td>-92.2</td>
<td>-5.4</td>
<td>89</td>
<td></td>
</tr>
<tr>
<td>Average SV</td>
<td>12.5</td>
<td>-23.0</td>
<td>-30.7</td>
<td>-2.7</td>
<td>44.5</td>
<td></td>
</tr>
</tbody>
</table>

Sum of averages is 0.6

Adjustment factor $\frac{Sum}{Period} = \frac{0.6}{5} = 0.12$

Adjusted SV $\begin{bmatrix} 12.38 \\ -23.12 \\ -30.82 \\ -2.82 \\ 44.38 \end{bmatrix}$ Sum is now 0
The seasonal variations are: Monday: 12 customers, Tuesday: -23 customers, Wednesday: -30 customers, Thursday: -3 customers, and Friday: 44 customers.

c) Forecasting using additive model

Average increase in trend \( = \frac{122 - 98}{11 - 1} = \frac{24}{10} = 2.4 \)

<table>
<thead>
<tr>
<th>Week 4</th>
<th>Trend, T</th>
<th>Seasonal variation, S</th>
<th>Forecast, T + S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>122 + 3 \times 2.4 = 129.2</td>
<td>12</td>
<td>141 customers</td>
</tr>
<tr>
<td>Tuesday</td>
<td>122 + 4 \times 2.4 = 131.6</td>
<td>-23</td>
<td>109 customers</td>
</tr>
</tbody>
</table>

d) Seasonal variation, SV, using the multiplicative model

Seasonal variations for multiplicative model are obtained by dividing the time series data values by the trend values i.e. \( S = \frac{Y}{T} \)

Table 16.22 calculation table, Trend and seasonal variation (multiplicative model)

<table>
<thead>
<tr>
<th>Week/Day</th>
<th>Data value, y</th>
<th>Moving totals (add in fives)</th>
<th>&quot;Trend&quot; Moving average (total divide by 5)</th>
<th>SV ( S = \frac{Y}{T} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>113</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Tue</td>
<td>72</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Wed</td>
<td>65</td>
<td>490</td>
<td>98</td>
<td>0.663</td>
</tr>
<tr>
<td>Thu</td>
<td>95</td>
<td>495</td>
<td>99</td>
<td>0.960</td>
</tr>
<tr>
<td>Fri</td>
<td>145</td>
<td>504</td>
<td>100.8</td>
<td>1.438</td>
</tr>
<tr>
<td>Week 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>118</td>
<td>516</td>
<td>103.2</td>
<td>1.143</td>
</tr>
<tr>
<td>Tue</td>
<td>81</td>
<td>531</td>
<td>106.2</td>
<td>0.763</td>
</tr>
<tr>
<td>Wed</td>
<td>77</td>
<td>546</td>
<td>109.2</td>
<td>0.705</td>
</tr>
<tr>
<td>Thu</td>
<td>110</td>
<td>557</td>
<td>111.4</td>
<td>0.987</td>
</tr>
<tr>
<td>Fri</td>
<td>160</td>
<td>576</td>
<td>115.2</td>
<td>1.389</td>
</tr>
<tr>
<td>Week 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>129</td>
<td>594</td>
<td>118.8</td>
<td>1.086</td>
</tr>
<tr>
<td>Tue</td>
<td>100</td>
<td>604</td>
<td>120.8</td>
<td>0.828</td>
</tr>
<tr>
<td>Wed</td>
<td>95</td>
<td>610</td>
<td>122</td>
<td>0.779</td>
</tr>
<tr>
<td>Thu</td>
<td>120</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Fri</td>
<td>166</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Averaging the SVs: We arrange in a table as before and average them to sum to 5 (the period)
### Table 16.23 Average seasonal variation – multiplicative model

<table>
<thead>
<tr>
<th>Week</th>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td>-</td>
<td>-</td>
<td>0.663</td>
<td>0.960</td>
<td>1.438</td>
</tr>
<tr>
<td>Week 2</td>
<td>1.143</td>
<td>0.763</td>
<td>0.705</td>
<td>0.987</td>
<td>1.389</td>
</tr>
<tr>
<td>Week 3</td>
<td>1.086</td>
<td>0.828</td>
<td>0.779</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Total</td>
<td>2.229</td>
<td>1.591</td>
<td>2.147</td>
<td>1.947</td>
<td>2.827</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Average SV</th>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.115</td>
<td>0.796</td>
<td>0.716</td>
<td>0.974</td>
<td>1.414</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Adjustment factor Period Sum = 5 Sum = 0.997</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplying each SV average by the adjustment factor.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Adjusted SV</th>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.112</td>
<td>0.793</td>
<td>0.714</td>
<td>0.971</td>
<td>1.410</td>
</tr>
</tbody>
</table>

The seasonal variations are:

- Monday: 1.112 (or 111.2%),
- Tuesday: 0.793 (or 79.3%),
- Wednesday: 0.714 (or 71.4%),
- Thursday: 0.971 (or 97.1%), and
- Friday: 1.41 (or 141%)

e) Forecasting using the multiplicative model

We find the trend projections (i.e. extrapolate trend values) as before. However, the forecasts are obtained by multiplying the trend estimates by the seasonal variations.

Average increase in trend = \( \frac{122 - 98}{11 - 1} = \frac{24}{10} = 2.4 \)

<table>
<thead>
<tr>
<th>Week 4</th>
<th>Trend, T</th>
<th>Seasonal variation, S</th>
<th>Forecast, T × S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>122 + 3 × 2.4 = 129.2</td>
<td>1.112</td>
<td>129.2 × 1.112 = 144 customers</td>
</tr>
<tr>
<td>Tuesday</td>
<td>122 + 4 × 2.4 = 131.6</td>
<td>0.793</td>
<td>131.6 × 0.793 = 104 customers</td>
</tr>
</tbody>
</table>

Note
Unlike the moving average method, if the least squares method is used all times point will have trend values. The procedures for obtaining seasonal variations and forecast are the same as in moving average method.
CHAPTER SUMMARY

This chapter has covered the following concepts:

- A time series is a collection of data over a period of time has four components namely: trend, cyclic, seasonal and random components.
- A trend is the long-term direction of the time series.
- The cyclic component is the fluctuation above and below the long-term trend line over a longer period of time.
- The seasonal component is the pattern in the time series over shorter periods of time due to effects of seasons. These patterns tend to repeat themselves from time to time.
- The random variations are unpredictable.
- The trend component of a times series can be obtained using either the semi-average, least squares or moving average methods. The semi-average and least squares methods assume pure linearity of the trend. The moving average method is used to smooth the trend in a time series.
- To obtain the seasonal variations, we used two models. If the differences between the largest and smallest values for each year (week) are roughly the same or do not follow a particular pattern, then additive model is applicable. If the differences are increasing for an upward trend or decreasing for a downward trend, then the multiplicative model is applicable.
- In the additive model the sum of the average seasonal variations must be zero. For the multiplicative models the average seasonal variations add up to the period of the time series.
- The seasonal variations are used to seasonally adjust data observations, and to forecast future activities, taking into account effects of season.

END OF CHAPTER EXERCISES

1. Briefly describe the components which make up a typical time series
2. The daily output of Chidziwitso Ltd over a four week period is shown in the table below:

<table>
<thead>
<tr>
<th>Week</th>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td>187</td>
<td>213</td>
<td>210</td>
<td>227</td>
<td>247</td>
</tr>
<tr>
<td>Week 2</td>
<td>207</td>
<td>218</td>
<td>215</td>
<td>234</td>
<td>256</td>
</tr>
<tr>
<td>Week 3</td>
<td>202</td>
<td>228</td>
<td>225</td>
<td>248</td>
<td>266</td>
</tr>
<tr>
<td>Week 4</td>
<td>208</td>
<td>255</td>
<td>245</td>
<td>257</td>
<td>278</td>
</tr>
</tbody>
</table>

a) Find the trend by five-point moving averages
b) Display on the same graph the actual data together with the trend figures
c) Determine the daily deviations from the trend and use these to determine the average adjusted daily variations

d) Forecast the daily output for the all days of week 5 to the nearest unit of production

3

a) Distinguish between the ‘additive model’ and the ‘multiplicative model’ in time-series analysis.
b) The following set of data represents a mining company’s quarterly production levels (y), in thousands of tonnes over 3 years:

<table>
<thead>
<tr>
<th>Year</th>
<th>Jan-Mar</th>
<th>Apr-Jun</th>
<th>Jul-Sep</th>
<th>Oct-Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>2011</td>
<td>20</td>
<td>50</td>
<td>40</td>
<td>100</td>
</tr>
<tr>
<td>2012</td>
<td>40</td>
<td>120</td>
<td>150</td>
<td>250</td>
</tr>
<tr>
<td>2013</td>
<td>100</td>
<td>220</td>
<td>280</td>
<td>450</td>
</tr>
</tbody>
</table>

i. Calculate a centered four-point moving average trend

ii. Using the multiplicative model and the trend estimated in (i), estimate the seasonal factors in each quarter (to 3 decimal places)

iii. Use the trend estimated in (i) and the seasonal factors estimated in (ii) to forecast the company’s production in all 4 quarters of 2014 (to the nearest whole numbers).

iv. Comment on the likely accuracy of your forecasts in (iii).

4 The consumer price index (CPI) for a village is shown quarterly over three years in following table.

<table>
<thead>
<tr>
<th>Year</th>
<th>Quarter</th>
<th>CPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>Jan-Mar</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>108</td>
</tr>
<tr>
<td></td>
<td>Oct-Dec</td>
<td>102</td>
</tr>
<tr>
<td>2011</td>
<td>Jan-Mar</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>110</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>112</td>
</tr>
<tr>
<td></td>
<td>Oct-Dec</td>
<td>106</td>
</tr>
<tr>
<td>2012</td>
<td>Jan-Mar</td>
<td>106</td>
</tr>
<tr>
<td></td>
<td>Apr-Jun</td>
<td>116</td>
</tr>
<tr>
<td></td>
<td>Jul-Sep</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td>Oct-Dec</td>
<td>110</td>
</tr>
</tbody>
</table>

a) Find the least squares linear trend line. Use the equation to find the trend values for each quarter for 2010 to 2012.
b) Using the additive model and the trend from part (a), estimate the average seasonal variations in each quarter (to two decimal places)
c) Deseasonalise the CPI values for 2011.

d) Use your results in (a) and (b) to forecast the CPI for the four quarters of 2013 (to the nearest whole number). Suggest two reasons why your forecast may not be reliable.

5 The planning department of Bata Shoe has developed the following least squares trend equation for sales, in thousands of pairs, based on five years quarterly data from 2009.

\[ Y = 3.30 + 1.75x \]

The following table gives the seasonal variations (SV) for each quarter.

<table>
<thead>
<tr>
<th>Quarter</th>
<th>Qtr 1</th>
<th>Qtr 2</th>
<th>Qtr 3</th>
<th>Qtr 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>1.10</td>
<td>1.20</td>
<td>0.80</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Forecast the sales for each quarter of 2014.

6 A shop is open every day except Fridays and Sundays. The number of customers visiting the shop each day for three weeks is show below

<table>
<thead>
<tr>
<th>Week</th>
<th>Day</th>
<th>Customers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Monday</td>
<td>160</td>
</tr>
<tr>
<td></td>
<td>Tuesday</td>
<td>170</td>
</tr>
<tr>
<td></td>
<td>Wednesday</td>
<td>140</td>
</tr>
<tr>
<td></td>
<td>Thursday</td>
<td>180</td>
</tr>
<tr>
<td></td>
<td>Saturday</td>
<td>280</td>
</tr>
<tr>
<td>2</td>
<td>Monday</td>
<td>170</td>
</tr>
<tr>
<td></td>
<td>Tuesday</td>
<td>180</td>
</tr>
<tr>
<td></td>
<td>Wednesday</td>
<td>140</td>
</tr>
<tr>
<td></td>
<td>Thursday</td>
<td>190</td>
</tr>
<tr>
<td></td>
<td>Saturday</td>
<td>300</td>
</tr>
<tr>
<td>3</td>
<td>Monday</td>
<td>200</td>
</tr>
<tr>
<td></td>
<td>Tuesday</td>
<td>210</td>
</tr>
<tr>
<td></td>
<td>Wednesday</td>
<td>180</td>
</tr>
<tr>
<td></td>
<td>Thursday</td>
<td>230</td>
</tr>
<tr>
<td></td>
<td>Saturday</td>
<td>340</td>
</tr>
</tbody>
</table>

a) Find a five-day moving average trend
b) Using the multiplicative model, estimate the seasonal factors for each of the five days.
c) Seasonally adjust the number of customer for the days from wee 1 to week 3.
d) Forecast the number of customers for each day of the fourth week
CHAPTER 17 INDEX NUMBERS

LEARNING OBJECTIVES

By the end of this chapter the student should be able to

i. Explain what an index number is
ii. Distinguish between base year and current year.
iii. Construct single item indices (price and quantity).
iv. Identify different types of index numbers.
v. Change the base of an index number
vi. Measure changes in economic data using indices
vii. Calculate the Laspeyres and Paasche Indices and explain the difference
viii. To adjust nominal money values into real terms (taking inflation into account)
ix. Explain how index numbers can be used in practice.

17.0 INTRODUCTION

In business, managers may be concerned with the way in which the values of most variables change over time: prices paid for raw materials; numbers of employees and customers, annual income and profits, etc. Index numbers are one way of describing such changes.

Index numbers were originally developed by economists for monitoring and comparing different groups of goods. It is necessary in business to be able to understand and manipulate the different published index series, and to construct your own index series.

17.1 DEFINITION

Index numbers are numbers which are used to measure changes in economic data over a period of time.

Index numbers measure the changing value of a variable over time in relation to its value at some fixed point in time, the base period, when it is given the value of 100.

Such indexes are often used to show overall changes in a market, an industry or the economy. For example, an accountant at a supermarket chain could construct an index of the chain’s own sales and compare it to the index of the volume of sales for the overall supermarket industry.

Examples of other economic data for which index numbers may be suitable include:

- Consumer prices
- Production levels
- Imports and exports
- Prices of shares

17.2 EXAMPLES OF INDEX NUMBERS (OR INDICES).

Several types of index numbers exist depending on the type of data whose changes are being measured. Examples include:

- Consumer index numbers
Production index numbers  
Quantity index numbers  
Wholesale index numbers  
Share price index numbers

A typical consumer price index (CPI) would appear as follows

<table>
<thead>
<tr>
<th>Years</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>100</td>
</tr>
<tr>
<td>2001</td>
<td>101</td>
</tr>
<tr>
<td>2002</td>
<td>109</td>
</tr>
<tr>
<td>2003</td>
<td>114</td>
</tr>
<tr>
<td>2004</td>
<td>121</td>
</tr>
<tr>
<td>2005</td>
<td>129</td>
</tr>
</tbody>
</table>

2000 = 100

Since the numbers are rising as it can be said that the data whose changes the index is measuring has an is rising. The index (CPI) is about consumer prices, it can be said that consumer prices are rising.

Note: The notation 2000 = 100 means that the base year (the reference point) is 2000.

17.3 CONSTRUCTING AN INDEX

Index for any time period n = \[ \frac{\text{value in current period}}{\text{value in base period}} \times 100 \]

Thus an index number is basically a ratio of two values one for the current period and the other at some reference point called base period (see below) and expresses as a percentage without showing the percentage sign.

17.4 SINGLE ITEM INDICES

Single item index numbers are index numbers that measure changes in data relating to a single item. The data could be price or quantity. A single item index is called a relative. A relative is actually the basic index.

If the single item index is about price of the item, then it is called a price relative. If it relates to quantities it is a quantity relative.

17.4.1 Price relatives

A price relative is the ratio of a current price to that of a base period.

If \( p_1 \) denotes the current price and \( p_0 \) the base period price

A price relative is calculated as: \( \frac{p_1}{p_0} \times 100 \)
Example 1

The average price of bread in Malawi for the past three years has been as follows

<table>
<thead>
<tr>
<th>Year</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price (MK)</td>
<td>120</td>
<td>150</td>
<td>210</td>
</tr>
</tbody>
</table>

Calculate the price relatives for 2010 and 2011 with 2009 as base year (i.e. 2000 = 100)

**Solution**

Table 17.2 Calculation of price relatives

<table>
<thead>
<tr>
<th>Year</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price (MK)</td>
<td>120</td>
<td>150</td>
<td>210</td>
</tr>
<tr>
<td>Price Relative</td>
<td>( \frac{P_1}{P_0} \times 100 )</td>
<td>( \frac{P_1}{P_0} \times 100 )</td>
<td>( \frac{P_1}{P_0} \times 100 )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{2009:} & \quad \frac{150}{120} \times 100 = 125.0 \\
\text{2010:} & \quad \frac{210}{210} \times 100 = 175.0 \\
\end{align*}
\]

The price relatives are:

<table>
<thead>
<tr>
<th>Year</th>
<th>Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>2009</td>
<td>100</td>
</tr>
<tr>
<td>2010</td>
<td>125.0</td>
</tr>
<tr>
<td>2011</td>
<td>175.0</td>
</tr>
</tbody>
</table>

17.4.2 Quantity Relatives

Like a price relative, a quantity relative is calculated as:

\[
\frac{Q_1}{Q_2} \times 100 \quad \text{Where} \quad Q_1 = \text{current quantity and} \quad Q_0 = \text{base period quantity}
\]

Example 2

KK Confectionaries Ltd bought the following quantities of flour for her baking.

<table>
<thead>
<tr>
<th>Month</th>
<th>Qty of flour (Kgs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>July</td>
<td>11</td>
</tr>
</tbody>
</table>

Calculate the quantity relatives with July = 100

Solution

Table 17.3 Calculation of quantity relative

<table>
<thead>
<tr>
<th>Month</th>
<th>July</th>
<th>Aug</th>
<th>Sept</th>
<th>Oct</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Q₀</td>
<td>Q₁</td>
<td>Q₁</td>
<td>Q₁</td>
</tr>
<tr>
<td>Quantity</td>
<td>11</td>
<td>8</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>Quantity Relative</td>
<td>Q₀ x 100</td>
<td>Q₁ x 100</td>
<td>Q₁ x 100</td>
<td>Q₀</td>
</tr>
<tr>
<td></td>
<td>150 x 100</td>
<td>210 x 100</td>
<td>120</td>
<td>120</td>
</tr>
<tr>
<td></td>
<td>100.0</td>
<td>72.7</td>
<td>109.1</td>
<td>145.5</td>
</tr>
</tbody>
</table>

Note that the drop in quantity of August is reflected in that the August relative is lower than that of July.

17.4.3 Finding values from Indexes

For index linked values (eg. salaries, prices of products), if we know all the index numbers and one of the index linked values we can find all the other values by scaling accordingly.

Example 3

Table 17.4 Monthly price index for an item:

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>June</th>
<th>July</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
<td>121</td>
<td>112</td>
<td>98</td>
<td>81</td>
<td>63</td>
<td>57</td>
<td>89</td>
<td>109</td>
<td>131</td>
<td>147</td>
<td>132</td>
<td>126</td>
</tr>
</tbody>
</table>

If the prices are index linked and if the price of a product in March is K240, what is its price in the other months?

Price in March is K240

⇒ Price in January is $K240 \times \frac{121}{98} = K296.30$

⇒ Price in February is $K240 \times \frac{112}{98} = K274.30$

For the other months, the prices are calculated in a similar way and we summarise them, to the nearest Kwacha, in the following table.

Table 17.5 Indices and estimated prices
17.4.4 Changing the base period

The choice of base period can be any convenient time - it is periodically adjusted.

It is usual to update the base period when:

- Any significant change which makes comparison with earlier figures meaningless.
- The numbers are growing so large that the size of a points change is many times that of a percentage change.

We need to know how to change a base period.

Published index numbers may include base changes: a series 470, 478, 485 continues 100, 103, 105, etc.

A common base year is needed before analysis as a continuous series can be undertaken.

In order to change from one index series to another we need values for both indexes in one period. The ratio of these two values forms the basis of any conversion between them.

**Formulae:**

\[
\text{New} = \text{Old} \times \frac{\text{New}}{\text{Old}} \quad \text{and} \quad \text{Old} = \text{New} \times \frac{\text{Old}}{\text{New}}
\]

**Example 4**

The advertising expenditure by a supermarket is index linked and is described by the following indexes:

Table 17.6 Indices with different bases

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index 1</td>
<td>100</td>
<td>138</td>
<td>162</td>
<td>196</td>
<td>220</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Index 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100</td>
<td>125</td>
<td>140</td>
</tr>
<tr>
<td>Adverts (K)</td>
<td></td>
<td>4860</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Complete each index series, then, given the expenditure for year 3, calculate the expenditure for the other years.

**Solution**

a) Base year for each index:

Index 1: Year 1 while Index 2: Year 5
b) **Missing values** for Index 1 and Index 2: (Both values known for year 5)

<table>
<thead>
<tr>
<th>Year</th>
<th>Index 1</th>
<th>Index 2</th>
<th>Values increased</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>138</td>
<td>162</td>
<td>196</td>
<td>220</td>
<td>275</td>
<td>308</td>
</tr>
<tr>
<td>Index 2</td>
<td>100</td>
<td>125</td>
<td>all values reduced</td>
<td>125</td>
<td>125</td>
<td>140</td>
<td>165</td>
</tr>
<tr>
<td>Adverts (£)</td>
<td>3000</td>
<td>4140</td>
<td>4860</td>
<td>5880</td>
<td>6600</td>
<td>8250</td>
<td>9240</td>
</tr>
</tbody>
</table>

Index 1 = \( \frac{220}{100} \times \frac{125}{100} = 275 \)

**Index 2 for Year 1**

Index 2 = \( \frac{100}{220} \times \frac{100}{200} = 45.45 \)

a) **Advertising expenditure** is calculated for all years from the known year using the ratio of the relevant index numbers from both series.

If advertising cost £4860 in year 3, that for:

Year 1 = \( \frac{4860}{162} \times \frac{100}{162} = 3000 \) or \( \frac{4860}{73.64} \times 45.45 = 3000 \)

Year 2 = \( \frac{4860}{162} \times \frac{138}{162} = 4140 \) or \( \frac{4860}{73.64} \times 62.73 = 4140 \)

**Table 17.7 Linked indices**

### 17.5 MULTI-ITEM INDICES

In the previous section, we dealt with single item indices (Relatives). In practice index numbers involve many items. For example a consumer price index measures changes in prices of consumer goods. Consumer goods may include various food stuffs, clothing, beverages or even transport services. The idea is that the changes in the prices of the items must be measured by a single series of numbers which means these prices, must be combined.

Techniques of combining the items include those where weights (see below) are used and those where weights are not used.

#### 17.5.1 Un-weighted Indices

An Un-weighted multi-item index can be calculated using any of the following methods.

i) The arithmetic mean of the price or quantity relatives

ii) Using the geometric mean of the relatives

iii) By calculation a simple aggregate index

We now look at each one of them using an example.

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Example 5

KK confectionaries Ltd produces and sells four sizes of party cakes: the mini, standard, family and super. The prices and quantities sold in the past 2 years are as follows:

Table 17.8 Price and quantities of selected commodities

<table>
<thead>
<tr>
<th>Item</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price (K)</td>
<td>Qty ('00)</td>
</tr>
<tr>
<td>Mini</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Standard</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>Family</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>Super</td>
<td>50</td>
<td>10</td>
</tr>
</tbody>
</table>

Calculate a multi-item price index for 2010 taking 2009 as base year using

a) The arithmetic mean of the price relatives
b) The geometric mean of the price relatives.
c) The simple aggregate index

Solution

Let \( Q_1 \) denote the current quantity and \( Q_0 \) denote the base period quantity.
Let \( P_1 \) denote the current price and \( P_0 \) denote the base period price.

Table 17.9 Calculation table (Price relatives)

<table>
<thead>
<tr>
<th>Item</th>
<th>P₀</th>
<th>Q₀</th>
<th>P₁</th>
<th>Q₁</th>
<th>Price relatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mini</td>
<td>10</td>
<td>20</td>
<td>12</td>
<td>15</td>
<td>( \frac{12}{10} \times 100 = 120 )</td>
</tr>
<tr>
<td>Standard</td>
<td>20</td>
<td>40</td>
<td>25</td>
<td>25</td>
<td>( \frac{25}{20} \times 100 = 125 )</td>
</tr>
<tr>
<td>Family</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>50</td>
<td>( \frac{30}{30} \times 100 = 100 )</td>
</tr>
<tr>
<td>Super</td>
<td>50</td>
<td>10</td>
<td>55</td>
<td>10</td>
<td>( \frac{55}{50} \times 100 = 110 )</td>
</tr>
</tbody>
</table>

a) The arithmetic mean of the price relatives is

\[
\frac{120 + 125 + 100 + 110}{4} = 113.75
\]

b) The geometric mean of the price relatives is

\[
\sqrt[4]{120 \times 125 \times 100 \times 110} = 113.34
\]

c) Simple aggregate index is

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17.5.2 Weighted indices

Concept of a weight

A weight is a figure that reflects the relative importance of a data value. Weights are important in that when data items (prices or quantities) are being combined in a calculation, the weights reflect the importance of that item. The larger the weight, the more the influence of that item will be in a sum.

Example 6

Jim and Zack are being assessed on the basis of their grades in Maths and Chichewa for a post of Accounts Assistant. There grades are as follows.

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Mathematics</th>
<th>Chichewa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jim</td>
<td>50%</td>
<td>95%</td>
</tr>
<tr>
<td>Zack</td>
<td>75%</td>
<td>55%</td>
</tr>
</tbody>
</table>

Work out a normal arithmetic mean.

a) Calculate a weighted average giving Mathematics a weight of 8 and Chichewa a weight of 2 (the meaning here being) Mathematics is 4 times as important as Chichewa in the accounts work).

b) Which is the appropriate way of selecting the candidates?

Solution

a) Table 17.10 Calculation table - Arithmetic mean

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Mathematics</th>
<th>Chichewa</th>
<th>Arithmetic mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jim</td>
<td>50%</td>
<td>95%</td>
<td>(50 + 95)/2 = 73%</td>
</tr>
<tr>
<td>Zack</td>
<td>75%</td>
<td>55%</td>
<td>(75 + 55)/2 = 65%</td>
</tr>
</tbody>
</table>

In this case Jim gets the job but indications are that he will struggle to perform as accounting requires numeracy skills indicated by Mathematics.
b) **Table 17.11 Calculation table Weighted mean**

<table>
<thead>
<tr>
<th>Grade</th>
<th>Maths</th>
<th>Chichewa</th>
<th>Weighted mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights</td>
<td>8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Jim</td>
<td>50%</td>
<td>95%</td>
<td>(50 x 8 + 95 x 2) / 8 + 2 = 59%</td>
</tr>
<tr>
<td>Zack</td>
<td>75%</td>
<td>55%</td>
<td>(50 x 8 + 95 x 2) / 8 + 2 = 71%</td>
</tr>
</tbody>
</table>

c) The weight of 8 has amplified the grade in Mathematics. As long as the weighting is acceptable this is the better way of selecting the candidates. Zack gets the Job.

**Example 7**

If the weights for KK confectionaries products are:

- Mini 6
- Standard 10
- Family 8
- Super 5

Calculate the 2010 index using:

a) a weighted mean of the price relatives

b) a weighted quantity index

**Solution**

**Table 17.12 Calculation table- Mean of Price relatives**

<table>
<thead>
<tr>
<th>Item</th>
<th>Price relatives</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mini</td>
<td>(12/10) x 100</td>
<td>120</td>
</tr>
<tr>
<td>Standard</td>
<td>(25/20) x 100</td>
<td>125</td>
</tr>
<tr>
<td>Family</td>
<td>(30/30) x 100</td>
<td>100</td>
</tr>
<tr>
<td>Super</td>
<td>(55/50) x 100</td>
<td>110</td>
</tr>
</tbody>
</table>

Index = 
\[
\frac{120 \times 6 + 125 \times 10 + 100 \times 8 + 110 \times 5}{6 + 10 + 8 + 5} = 114.48
\]

a) **A weighted index**

A weighted price index is calculated as follows:

\[
I = \frac{\sum P_i W_i}{\sum P_i W_i} \times 100, \text{ where } W_i \text{ is a weight}
\]
A weighted quantity index is calculated as

$$\frac{\sum Q_n W}{\sum Q_o W} \times 100$$

Table 17.13  Weighted indices

<table>
<thead>
<tr>
<th>Item</th>
<th>2009</th>
<th>2010</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P0</td>
<td>Q0</td>
<td>Pn</td>
</tr>
<tr>
<td>Mini</td>
<td>10</td>
<td>20</td>
<td>12</td>
</tr>
<tr>
<td>Standard</td>
<td>20</td>
<td>40</td>
<td>25</td>
</tr>
<tr>
<td>Family</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>Super</td>
<td>50</td>
<td>10</td>
<td>55</td>
</tr>
</tbody>
</table>

The weighted price index = $$\frac{(12 \times 6 + 25 \times 10 + 30 \times 8 + 55 \times 5)}{(10 \times 6 + 20 \times 10 + 30 \times 8 + 50 \times 5)} \times 100$$

$$= \frac{837}{750} \times 100$$

$$= 111.6$$

b) A weighted quantity index is calculated in a similar manner

**Determination of weights**

The weights cited in these examples may have resulted from an agreement or from an “educated guess”. In reality when a price index is being calculated, the quantities become the weights and when a quantity index is being calculated it is the prices that become the weights.

Prices and quantities occur both in the base year as well as current year. The question is which prices or quantities are used as weights. **Base or the current year quantities or prices may be used as weights.**

> *When base year prices or quantities are used as weights the index is called a LASPEYRES INDEX. While if current year weights are used, the index is called the PAASCHE index.*

**17.5.3 The Laspeyres index**

The Laspeyres Price index is calculated as follows:

$$\frac{\sum P_n Q_0}{\sum P_0 Q_0} \times 100$$, where base year Quantities $Q_0$ are the weights
The Laspeyres Quantity is calculated as follows:

\[ \frac{\sum Q_n P_0}{\sum Q_n} \times 100, \] where now base year prices \( P_0 \) are the weights.

**Example 8**

Use the KK Confectionaries data below to calculate the Laspeyres price and quantity index for 2010.

<table>
<thead>
<tr>
<th>Item</th>
<th>2009 Price (K)</th>
<th>Qty ('00)</th>
<th>2010 Price (K)</th>
<th>Qty ('00)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mini</td>
<td>10</td>
<td>20</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>Standard</td>
<td>20</td>
<td>40</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>Family</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>50</td>
</tr>
<tr>
<td>Super</td>
<td>50</td>
<td>10</td>
<td>55</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item</th>
<th>Po</th>
<th>Qo</th>
<th>Pn</th>
<th>Qn</th>
<th>PoQo</th>
<th>PnQo</th>
<th>QnPo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mini</td>
<td>10</td>
<td>20</td>
<td>12</td>
<td>15</td>
<td>200</td>
<td>240</td>
<td>150</td>
</tr>
<tr>
<td>Standard</td>
<td>20</td>
<td>40</td>
<td>25</td>
<td>25</td>
<td>800</td>
<td>1000</td>
<td>500</td>
</tr>
<tr>
<td>Family</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>50</td>
<td>900</td>
<td>900</td>
<td>1500</td>
</tr>
<tr>
<td>Super</td>
<td>50</td>
<td>10</td>
<td>55</td>
<td>10</td>
<td>500</td>
<td>550</td>
<td>500</td>
</tr>
</tbody>
</table>

**Solution**

Table 17.15 Laspeyres index

<table>
<thead>
<tr>
<th>Item</th>
<th>Po</th>
<th>Qo</th>
<th>Pn</th>
<th>Qn</th>
<th>PoQo</th>
<th>PnQo</th>
<th>QnPo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mini</td>
<td>10</td>
<td>20</td>
<td>12</td>
<td>15</td>
<td>200</td>
<td>240</td>
<td>150</td>
</tr>
<tr>
<td>Standard</td>
<td>20</td>
<td>40</td>
<td>25</td>
<td>25</td>
<td>800</td>
<td>1000</td>
<td>500</td>
</tr>
<tr>
<td>Family</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>50</td>
<td>900</td>
<td>900</td>
<td>1500</td>
</tr>
<tr>
<td>Super</td>
<td>50</td>
<td>10</td>
<td>55</td>
<td>10</td>
<td>500</td>
<td>550</td>
<td>500</td>
</tr>
</tbody>
</table>

\[ \text{a) The Laspeyres price index} \quad \frac{2690 \times 100}{2400} = 112.08 \]

\[ \text{b) The Laspeyres quantity index} \quad \frac{2650 \times 100}{2400} = 110.42 \]

**17.5.4 The Paasche index**

As stated in 17.5.3 a Paasche index is an index which uses current year weights. Like the case of Laspeyres it is one can calculate a price or a quantity index.

\[ \text{The Paasche Price index} \quad = \frac{\sum P_n Q_n \times 100}{\sum P_n Q_n} \]

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The Paasche Quantity index \[= \frac{\sum P_n Q_n}{\sum P_0 Q_0} \times 100\]

**Example 9**

Using the same figures relating to KK confectionaries, calculate the Paasche price and quantity index for 2010.

**Solution**

Table 17.16 Paasche Index

<table>
<thead>
<tr>
<th>Item</th>
<th>Po</th>
<th>Qo</th>
<th>Pn</th>
<th>Qn</th>
<th>PoQn</th>
<th>PnQn</th>
<th>QnPo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mini</td>
<td>10</td>
<td>20</td>
<td>12</td>
<td>15</td>
<td>150</td>
<td>180</td>
<td>240</td>
</tr>
<tr>
<td>Standard</td>
<td>20</td>
<td>40</td>
<td>25</td>
<td>25</td>
<td>500</td>
<td>625</td>
<td>1000</td>
</tr>
<tr>
<td>Family</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>50</td>
<td>1500</td>
<td>1500</td>
<td>900</td>
</tr>
<tr>
<td>Super</td>
<td>50</td>
<td>10</td>
<td>55</td>
<td>10</td>
<td>500</td>
<td>550</td>
<td>550</td>
</tr>
<tr>
<td>Totals</td>
<td>2650</td>
<td>2855</td>
<td>2690</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) The Paasche price index \[= \frac{2855}{2690} \times 100 = 107.74\]

b) The Paasche quantity index \[= \frac{2855}{2650} \times 100 = 106.13\]

**17.6 USES OF INDEX NUMBERS**

Index numbers are very useful in measuring the relative changes in the value of money. They are very helpful for the guidance and formulation of economic policies. Index numbers of imports, exports, wages, and employment and population importance cannot be ignored.

The following are some of the main uses of index numbers:

a) **Index numbers act as economic barometers**

A barometer is an instrument that is used to measure atmospheric pressure. Index numbers are used to feel the pressure of the economic and business behaviour, as well as to measure ups and downs in the general economic condition of a country. For example, the composite index number of indexes of prices, industrial output, foreign exchange reserves, and bank deposits, could act as an economic barometer.
b) Importance for governments

In the industrialized world, an index number such as the Consumer Price Index (CPI) is the most widely used measure of inflation and is sometimes viewed as an indicator of the effectiveness of government economic policy. It provides information about price changes in a country’s economy to government, business, labor, and private citizens and is used by them as a guide to making economic decisions. Governments use trends in the CPI to aid in formulating fiscal and monetary policies.

c) Consumption standard

If we want to know the true consumption standard of a class in a locality we can compute the consumption index number.

d) Fixation of wages

The money wages can be revised according to the proportionate change in the cost of living. The cost of living index number guides the government and the executives for the fixation and revision of wages. Employees also use index numbers, such as CPI, for wage negotiation.

f) Importance for the producer

Price index numbers may provide an indication to the producer whether to increase or reduce production or if price level is rising it may suggest increasing profits.

g) Analysis of industry

If we want to judge the prospects of manufacturing concern the investment index number can be constructed, to know the net yield of the industrial sector.

h) Comparison of developed and under developed Countries

International price index number can be used for comparing the general level of prices in the developed and under developed countries.

i) Efficiency of labour

In order to check the efficiency and per capita output of labour, index can be used to show this can be shown by index number. Promotion and salary can be also considered keeping in view the index number.

j) Measures to remove inequality of income

Index numbers of wholesale prices also indicate about the regional disparity. So different measures can be taken for the proper distribution of wealth and stabilizing of prices.
CHAPTER SUMMARY

This chapter has covered the following under index numbers

- In general, index numbers provide a measure of change of a phenomenon from one time point, referred to as base time, to another time point called current time.
- In particular, a quantity index shows how a quantity changes over a period of time while a price index shows how the price of a product changes over a period of time.
- The chapter illustrated how an index is constructed and distinguished single item indices and multi-item indices.
- In particular, the chapter introduced the idea of weighted indices. Two popular weighted indices were explored: Laspeyres and Paasche index.
- When base year prices or quantities are used as weights the index is called a Laspeyres Index. While if current year weights are used, the index is called the Paasche index.

END OF CHAPTER EXERCISES

1. a) What is an index number?
   b) Give four examples of index numbers.

2. a) Explain why it may be necessary to change the base in index calculations.
   b) Describe any three uses of index numbers.

3. A company buys four products with the following characteristics:

<table>
<thead>
<tr>
<th>Items</th>
<th>Number of units bought</th>
<th>Price paid per unit (£)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Year 1</td>
<td>Year 2</td>
</tr>
<tr>
<td>A</td>
<td>20</td>
<td>24</td>
</tr>
<tr>
<td>B</td>
<td>55</td>
<td>51</td>
</tr>
<tr>
<td>C</td>
<td>63</td>
<td>84</td>
</tr>
<tr>
<td>D</td>
<td>28</td>
<td>34</td>
</tr>
</tbody>
</table>

   a) Find the simple price indexes for the products for year 2 using year 1 as the base year.

   b) Find the simple aggregate index for year 2 using year 1 as the base year.

   c) Find the base-weighted aggregate index, the Laspeyres index, for year 2 using year 1 as the base year.

   d) Find the current period-weighted aggregate index, the Paasche index, for year 2 using year 1 as the base year.
4. a) Change the base of the following index from year 2003 to year 2005:

<table>
<thead>
<tr>
<th>Year</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>2003</td>
<td>100</td>
</tr>
<tr>
<td>2004</td>
<td>116</td>
</tr>
<tr>
<td>2005</td>
<td>120</td>
</tr>
<tr>
<td>2006</td>
<td>130</td>
</tr>
<tr>
<td>2007</td>
<td>145</td>
</tr>
</tbody>
</table>

(b) A factory produces 3 types of agricultural equipment: tractors, ploughs and balers. The following table shows the prices of the 3 items and the quantities produced for the 2 years, 2008 and 2009:

<table>
<thead>
<tr>
<th>Year</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price per Unit (Km)</td>
<td>Quantity</td>
</tr>
<tr>
<td>Tractor</td>
<td>45</td>
<td>300</td>
</tr>
<tr>
<td>Plough</td>
<td>7</td>
<td>500</td>
</tr>
<tr>
<td>Baler</td>
<td>3</td>
<td>200</td>
</tr>
</tbody>
</table>

Required

(i) Calculate an unweighted aggregate price index for the factory’s agricultural equipment for 2009, using 2008 as the base year.

(ii) Calculate an unweighted geometric mean of price relatives for 2009, using 2008 as the base year.

(iii) Construct a Laspeyres price index for 2009, using 2008 as the base year.

(iv) Construct a Paasche price index for 2009, using 2008 as the base year.

5. The following information was recorded on the prices and consumption of tea, coffee and chocolate drinks:

<table>
<thead>
<tr>
<th>Drink</th>
<th>Year 0 Price</th>
<th>Year 0 Qty</th>
<th>Year 1 Price</th>
<th>Year 1 Qty</th>
<th>Year 2 Price</th>
<th>Year 2 Qty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tea</td>
<td>8</td>
<td>15</td>
<td>12</td>
<td>12</td>
<td>16</td>
<td>10</td>
</tr>
<tr>
<td>Coffee</td>
<td>15</td>
<td>3</td>
<td>17</td>
<td>3</td>
<td>18</td>
<td>4</td>
</tr>
<tr>
<td>Chocolate</td>
<td>22</td>
<td>1</td>
<td>23</td>
<td>3</td>
<td>24</td>
<td>5</td>
</tr>
</tbody>
</table>

Suppose the price is taken as the average amount paid in Kwacha for a cup and the quantity as the average number of cups drunk per person per week.

Required

a) Using year 0 as the base year, determine the simple aggregate price index for

i) Year 1 and

ii) Year 2

b) using year 0 as base year, determine the average price relatives, for

i) Year 1 and
ii) Year 2

Calculate the Laspeyres price index for year 1 only using year 0 as base year

(a) The following table shows an index of the annual number of advertisements placed by an organization in the press and the index of the number of organizations products sold per annum.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Advert placed (1990=100)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value of sales (1980=100)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1988</td>
<td>90</td>
<td>96</td>
<td>100</td>
<td>115</td>
<td>128</td>
<td>140</td>
<td>160</td>
</tr>
<tr>
<td>1989</td>
<td>340</td>
<td>347</td>
<td>355</td>
<td>420</td>
<td>472</td>
<td>515</td>
<td>572</td>
</tr>
</tbody>
</table>

Required:

Rebase the index of volume of sales to 1990 and compare the two sets of relatives.

(b) The following table shows the quantities sold and total revenue on three key food items (P, Q, and R) sold by a chain of supermarkets between 2007 and 2009.

<table>
<thead>
<tr>
<th>ITEM</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Quantity</td>
<td>Total revenue (K'm)</td>
<td>Quantity</td>
</tr>
<tr>
<td>P</td>
<td>2250 kg</td>
<td>2.50</td>
<td>2500KG</td>
</tr>
<tr>
<td>Q</td>
<td>12000 litres</td>
<td>12.15</td>
<td>14500 litres</td>
</tr>
<tr>
<td>R</td>
<td>1250 items</td>
<td>5.25</td>
<td>14500 litres</td>
</tr>
</tbody>
</table>

Required:

(i) Construct a Laspeyre price index for the food items for 2008 and 2009 using 2007 as base year.

(ii) Construct a Paasche quantity index for the food items for 2008 and 2009 using 2007 as base year

Describe four uses of index numbers in practice.
CHAPTER 18  FINANCIAL MATHEMATICS

LEARNING OBJECTIVES

By the end of this chapter, the student should be able to:

i. Distinguish between simple and compound interest
ii. Calculate interest, principal or period in given various combinations of parameters
iii. Should be able to compound at shorter periods and calculate the effective interest rate
iv. Describe the different techniques of depreciation.
v. Depreciate an asset using the various depreciation techniques
vi. Appraise an investment using pay back period
vii. Appraise an investment using Net Present Values
viii. Appraise an investment using Internal Rate of Return
ix. Compare the various techniques of investment appraisal
x. Calculate the maturity value of an annuity
xi. Calculate the fixed equal payment of annuity given the maturity value
xii. Define a sinking fund
xiii. Describe amortisation as a method of debt repayment

18.0 INTRODUCTION

Financial Mathematics is a collection of mathematical techniques that find applications in Finance related areas. This chapter looks at:

- Basic calculations on interests
- Investment decisions
- Calculation issues in asset management (e.g. of values of assets and depreciation)
- Loan issues

This text considers basic mathematical techniques useful in many computations, looks at the concepts of interest, depreciation, capital appraisal and annuities.

18.1 INTEREST

Introduction

In the context of finance, interest is amount one earns when a sum of money is invested in a bank account or otherwise. The term can also be used to mean what is paid over and above a sum of money one borrows. The amount which is borrowed or invested is called the principal.

Either way there are two ways of dealing with (calculating) interest. These are:

a) Simple interest
b) Compound interest

18.1.1 Simple interest

Simple interest is interest calculated on a fixed principal amount.

For a principal P, invested at a rate of R% per annum, over T years, simple interest I is given by

\[ I = P \times R \times T \]
Example 1

How much interest would K10,000 earn at 8% per annum simple interest over 15 years?

Solution:

\[ I = \frac{PTR}{100} \]

\[ = \frac{K10,000 \times 15 \times 8}{100} \]

\[ = 12,000 \]

Example 2

Jamadi wants to earn K500,000 in interest so that she can have enough money to buy a good used car. She puts K800,000 into an account that earns 6.5% p.a. simple interest. How long will she need to leave her money in the account to earn the K500,000 interest?

Solution

The question is about obtaining the time (T) required for the principal to earn interest of K500,000.

Making T subject of the formula \( I = \frac{PTR}{100} \), we obtain

\[ T = \frac{I \times 100}{P \times R} \]

\[ = \frac{K500,000 \times 100}{K800,000 \times 6.5} \]

\[ = 9.6 \text{ years} \]

Example 3

A man invests 10,000 in a bank account for 3 years at 5% simple interest. After the 3 years he is enticed to put all the money back into the account because the bank has offeres him 10% p.a. simple interest. If he does not withdraw any money, calculate the total amount he will have in the account after a further 3 years.

Solution

This is a problem of calculating interest two times:

Accrued amount (A) after the first 3 years:

\[ A = P + \text{Interest} \]

\[ = P + \frac{PTR}{100} \]

\[ = K10,000 + \frac{K10,000 \times 3 \times 5}{100} \]

\[ = K11,500 \]
Accrued amount after a further 3 years:

\[ A = K11,500 + \frac{K11,500 \times 3 \times 10}{100} = K14,500 \]

### 18.1.2 Compound interest

While simple interest calculates the interest on a fixed principal period after period, compound interest arises when interest earned in one period is added to the principal, so that the interest that has been added also earns interest.

**Example 4**

Calculate total interest and the amount in the account earned when K1000 is invested at 10% compound interest for 4 years.

<table>
<thead>
<tr>
<th>Year</th>
<th>Principal</th>
<th>Rate</th>
<th>Interest</th>
<th>Amount in Account</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
<td>0.1</td>
<td>100</td>
<td>1100</td>
</tr>
<tr>
<td>1</td>
<td>1100</td>
<td>0.1</td>
<td>110</td>
<td>1210</td>
</tr>
<tr>
<td>2</td>
<td>1210</td>
<td>0.1</td>
<td>121</td>
<td>1331</td>
</tr>
<tr>
<td>3</td>
<td>1331</td>
<td>0.1</td>
<td>133.1</td>
<td>1464.1</td>
</tr>
<tr>
<td>4</td>
<td>1464.1</td>
<td>0.1</td>
<td>146.41</td>
<td>1610.51</td>
</tr>
</tbody>
</table>

**Compound interest formula and calculation**

Assuming a sum \( P \) is invested in an account at a rate of \( r\% \) for \( t \) years, the accrued (sometimes called the terminal) amount \( A \) in the account after \( t \) years is given by:

\[ A = P \left(1 + \frac{r}{100}\right)^t \]

**Example 5**

K300,000 is invested for 4 years at 10% compound interest. Find the terminal value and interest gained after the 4 years.

**Solution**

Terminal value:

\[ A = P \left(1 + \frac{r}{100}\right)^t \]
Example 6
Kuganizira Ltd sells company vehicles to the officers using the vehicles when the car is 4 years old. The car is sold to the employee at 10% the purchase cost. Charles is using one such vehicle which cost the company Mk8,500,000 to buy and he is looking forward to buying it in three year from now. In order not to miss the chance, he has been advised to set aside some money for the purpose. How much should he invest into an account now to buy the vehicle if the fund he is to use pays 12% compound interest per annum.

Solution.
The problem is about finding the principal.

\[ Purchase \, Price = K8,500,000 \times \frac{10}{100} \]
\[ = K850,000 \]

\[ A = P(1 + \frac{r}{100})^t \]

Making \( P \) subject of the equation we obtain

\[ P = \frac{A}{(1 + \frac{r}{100})^t} \]
\[ = \frac{K850,000}{(1 + \frac{12}{100})^3} \]
\[ = K605,013.21 \]

Charles should invest K605,013.20 now.

Example 7
Ethel left K150,000 in her account at the bank, while going on study in the UK two years ago. She has come back and has found a total amount of K174,960.00. What was the rate of interest assuming the rate never changed in the period.
Solution

\[ A = P \left(1 + \frac{r}{100}\right)^t \]

\[ \frac{r}{100} = \frac{t}{\sqrt[174.960\%]{A - P} - 1} \]

\[ = \sqrt[150,000]{174.960\% - 1} \]

Rate of interest = 8%

**Example 8**

Wonga has identified a fund which pays a good 14% compound interest. She has K450,000 which she can invest. For how long should she invest the money to realise a total of K900,000 to buy a dream plot.

Solution:
The problem concerns working out the period \( t \)

\[ A = P \left(1 + \frac{r}{100}\right)^t \]

\[ \log \left(\frac{A}{P}\right) = t \log \left(1 + \frac{r}{100}\right) \]

\[ \log \left(\frac{900,000}{450,000}\right) = t \times \log \left(1 + \frac{14}{100}\right) \]

\[ t = \frac{\log(2)}{\log(1.14)} \]

= 5.3 years

**Compounding at shorter periods**

Apart from the annual basis, compounding can be at shorter periods like half-year, quarterly, monthly, weekly, or even daily.

When compounding is done more frequently than once a year, it earns even higher interest amounts. As it will be demonstrated in the example below, the shorter the compounding period in the year, the larger the amount relatively.

Note:

In general if an amount \( P \) is invested under compound interest which is done \( n \) times in year at a rate of \( r\% \) per annum, over \( t \) years, then the accrued amount is given by

\[ A = P \left(1 + \frac{r}{100n}\right)^{nt} \]
Example 9

300 000 is invested at 12% per annum. Find the amount at the end of 4 years if compounding is:

a) annually
b) ½ year
c) quarterly
d) monthly

Solution:

a) \[ A = P \left(1 + \frac{r}{100}\right)^t \]
\[ = K300,000 (1.12)^4 \]
\[ = K472,055.81 \]

b) \[ A = K300,000 \left(1 + \frac{\frac{12}{2}}{100}\right)^{2 \times 4} \]
\[ = K300,000 (1.06)^8 \]
\[ = K478,154.42 \]

c) \[ A = K300,000 (1.03)^{16} \]
\[ = K481,411.93 \]

d) \[ A = K300,000 (1.01)^{48} \]
\[ = K483,667.82 \]

The effective rate of interest (ERI)

Compound interest results in a higher terminal value because of the fact that interest earns further interest when it is added back to the principal. The given annual rate is called the nominal rate and the rate final rate resulting from compounding is called the effective rate of interest sometimes also called the Actual percentage Rate (APR).

Example 10

Given a nominal rate of \( r = 12\% \) per annum, find the effective interest rate if compounding is done

a) Quarterly
b) Monthly for one year

Solution

a) Annual rate \( = 12\% \)
Quarterly rate \( = \frac{12}{4} \% = 3\% \)

Effective rate \( = [(1.03)^4 - 1]\% = 12.55\% \)

When compounding monthly  
Monthly rate \( = \frac{12}{12} \% = 1\% \)

Effective rate \( = [(1.01)^{12} - 1]\% = 12.68\% \)
18.2 DEPRECIATION

Any asset purchased and is being used, will get worn out or lose its value at a certain rate during its lifetime. In business, depreciation is the term used to mean the part or proportion of the asset being consumed away or the loss in value. For example a bus bought by Axa now at say K20,000,000 will be worth only 12,000,000 in 3 years time due to wear and tear. The 8,000,000 decline in value is the depreciation over the 3 years.

Note that when depreciation is allowed for (charged), the value of the asset or loan amount declines. The resulting value is called The Net Book Value (NBV).

There are several techniques which can be used to calculate depreciation. These include

i. Straight line depreciation
ii. Sum of digits
iii. Reducing balance

18.2.1 Straight line depreciation

Straight line depreciation is when the value of an asset is reduced by fixed equal amounts over the life of the asset.

The amount of depreciation charged each year under straight line depreciation, also called the annual depreciation value, is calculated as follows

\[
\text{Depreciation Charge} = \frac{\text{Original Book Value} - \text{Scrap Value}}{\text{Useful Life (yrs) of Asset}}
\]

Scrap value is the residual value of an asset after its useful period. It is usually measured by how much it can fetch if sold after its useful life.

Example 11

A machine needs to be depreciated from K25,000 to K5000 over a period of 5 years using straight line method. Find:

a) Depreciation charge per annum
b) The net book value at the end of 3 years

Solution:

a) \[
\text{Depreciation Charge} = \frac{\text{Original Book Value} - \text{Scrap Value}}{\text{Useful Life (yrs) of Asset}} = \frac{25,000 - 5000}{5} = K4000.00 \text{ per annum}
\]

b) Net book value at the end of year 3:

\[
\text{NBV} = \text{Original Book Value} - (\text{Depreciation charge p.a. } \times \text{number of years of depreciation})
\]

\[
= 25,000 - 4000 \times 3
= 25,000 - 12,000
= K13,000.00
\]
18.2.2 Sum of digits

The “sum of digits” technique of calculating depreciation uses the digits of the years (i.e. year 1, 2, 3, 4, etc) of the useful life of the asset in reverse as weights to calculate depreciation charge for the years. If the asset is to be depreciated over 3 years, the first year gets a weight of 3, the second 2 and the last gets 1. This way depreciation is high in the early years of an asset.

Example 12

Based on example 11 above calculate the depreciation charges for each year and then calculate the NBV.

Solution:

<table>
<thead>
<tr>
<th>Year</th>
<th>Weight</th>
<th>Dep Charge for the year</th>
<th>NBV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>(5/15) × 20,000 = 6,666.67</td>
<td>25000 - 6666.67 = 18,333.33</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(4/15) × 20,000 = 5,333.33</td>
<td>18,333.33 - 5,333.33 = 13,000.00</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(3/15) × 20,000 = 4,000.00</td>
<td>9,000.00</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>(2/15) × 20,000 = 2,666.67</td>
<td>6,333.33</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>(1/15) × 20,000 = 1,333.33</td>
<td>5,000.00</td>
</tr>
</tbody>
</table>

18.2.3 Reducing balance method

Under reducing balance method depreciation for each year is a proportion of the net book value as at the beginning of that year. The proportion or rate remains fixed unless deliberately changed.

For an asset with an Original Book Value B, depreciating at a rate of r%, the depreciated value or NBV of the asset at the end of the $n^{th}$ year is given by:

\[ NBV = B \left(1 - \frac{r}{100}\right)^n \]

And the depreciation charge D for the $n^{th}$ year is

\[ D = B \times \frac{r}{100} \times \left(1 - \frac{r}{100}\right)^{n-1} \]

Example 13

A main frame computer costing K220,000 will depreciate to a scrap value of K12000 in 5 years.

a) If reducing balance method is used find the depreciation rate?
b) What is the book value of the computer at the end of the 3rd year? What is the depreciation charge for the year?

c) How much more would the book value be at the end of the 3rd year if we use straight line method?

Solution:

\[ NBV = B \left( 1 - \frac{r}{100} \right)^n \]

\[ K12,000 = 220,000 \left( 1 - \frac{r}{100} \right)^5 \]

\[ r = \left( 1 - \frac{12000}{22000} \right) \times 100 = 44.11 \]

Discounting rate = 44.11%

b) At the end of the third year:

\[ NBV = K220,000 \left( 1 - \frac{44.11}{100} \right)^3 = K38,408.29 \]

c) If straight line had been used

d) Depreciation Charge = \[ \frac{K220,000 - K12,000}{5} \]

At the end of the third year \( NBV = K220,000 - 3 \times K41,600 = K95,200 \)

The NBV would have been \( K95,200 - K38,408.29 = 56791.71 \) more if straight line method had been used.

18.3 DISCOUNTING

Discounting is connected to the concept of **time value of money**. Time value of money refers to the fact that the same amount of money occurring at different time points will have different values or will purchase different quantities of similar goods due to among many factors, inflation. Specifically, Money loses value with time. The idea that money available at the present time is worth more than the same amount in the future due to its potential earning capacity makes it difficult to compare sums of money realised at different times. This concept is referred to as time value of money. Discounting helps to determine this time value of money.

**Definition**

Discounting is the process of measuring how much a future sum of money is worth now. In other words, discounting is the process for finding the present value of money.

Note that discounting is the reverse of compounding.
Given a future sum A, discounted a rate r% for n periods, its Present Value is given by:

\[ PV = \frac{A}{\left(1 + \frac{r}{100}\right)^n} \]

**Example 14**
Mrs Kalima expects to earn K500,000 from her maize seed project this time next year. Find the present value of the earnings if the rate of interest on the market is 10%

Solution

**Present Value:**

\[ PV = \frac{A}{\left(1 + \frac{r}{100}\right)^n} = \frac{K500,000}{\left(1 + \frac{10}{100}\right)^1} = K454,545.45 \]

**Example 15**
Mr. Kalima intends to sell his pickup 3 years from now for about K700,000. Advise him as to how much that sum is worth now if the rate of interest is 8%

**Present Value:**

\[ PV = \frac{K700,000}{\left(1 + \frac{8}{100}\right)^3} = K555,682.57 \]

**Example 16**
A departmental store advertises goods at K70,000.00 deposit and 3 further equal annual payments of K50,000.00. If the discount rate is 8%, calculate the present value of the goods.

Solution

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow (payments)</td>
<td>70,000</td>
<td>50,000</td>
<td>50,000</td>
<td>50,000</td>
</tr>
</tbody>
</table>

\[ PV = K70,000 + \frac{K50,000}{(1 + \frac{10}{100})^1} + \frac{K50,000}{(1 + \frac{10}{100})^2} + \frac{K50,000}{(1 + \frac{10}{100})^3} = K198,854.85 \]

**Discounting Factors**
A discounting factor is the present value of one unit currency (One Kwacha in our case)

i.e. Discounting factor \( = \frac{1}{\left(1 + \frac{r}{100}\right)^n} \)
Thus \( PV = A \times \text{Discounting factor} \)

**Example 17**

Use discounting factors to calculate the Present values of the cash flow in example 30 above.

**Solution**

Table 18.3 Calculation table – Present values

<table>
<thead>
<tr>
<th>Year</th>
<th>Net Cash flow (K)</th>
<th>Discounting factor (DF 8%)</th>
<th>Present Value (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>70,000</td>
<td>( \left(1 + \frac{8}{100}\right)^0 = 1.0000 )</td>
<td>70,000</td>
</tr>
<tr>
<td>1</td>
<td>50,000</td>
<td>( \left(1 + \frac{8}{100}\right)^1 = 0.9259 )</td>
<td>46,296.30</td>
</tr>
<tr>
<td>2</td>
<td>50,000</td>
<td>( \left(1 + \frac{8}{100}\right)^2 = 0.8573 )</td>
<td>42,866.94</td>
</tr>
<tr>
<td>3</td>
<td>50,000</td>
<td>( \left(1 + \frac{8}{100}\right)^3 = 0.7938 )</td>
<td>39,691.61</td>
</tr>
</tbody>
</table>

While discounting factors can be calculated on a calculator using the formula presented above, tables of already calculated discounting factors are available as shown below. Note that the factors are calculated to three decimal places only.
### Table 18.4  
Discounting factors

<table>
<thead>
<tr>
<th>RATE</th>
<th>Value of K1 in year n</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0%</td>
<td>0.962 0.925 0.889 0.855 0.822 0.790 0.760 0.731 0.703 0.676 0.650</td>
</tr>
<tr>
<td>5.0%</td>
<td>0.952 0.907 0.864 0.823 0.784 0.746 0.711 0.677 0.645 0.614 0.585</td>
</tr>
<tr>
<td>6.0%</td>
<td>0.943 0.890 0.840 0.792 0.747 0.705 0.665 0.627 0.592 0.558 0.527</td>
</tr>
<tr>
<td>7.0%</td>
<td>0.935 0.873 0.816 0.763 0.713 0.666 0.623 0.582 0.544 0.508 0.475</td>
</tr>
<tr>
<td>8.0%</td>
<td>0.926 0.857 0.794 0.735 0.681 0.630 0.583 0.540 0.500 0.463 0.429</td>
</tr>
<tr>
<td>9.0%</td>
<td>0.917 0.842 0.772 0.708 0.650 0.596 0.547 0.502 0.465 0.422 0.388</td>
</tr>
<tr>
<td>10.0%</td>
<td>0.909 0.826 0.751 0.683 0.621 0.564 0.513 0.467 0.424 0.386 0.350</td>
</tr>
<tr>
<td>11.0%</td>
<td>0.901 0.812 0.731 0.659 0.593 0.535 0.482 0.434 0.391 0.352 0.317</td>
</tr>
<tr>
<td>12.0%</td>
<td>0.893 0.797 0.712 0.636 0.567 0.507 0.452 0.404 0.361 0.322 0.287</td>
</tr>
<tr>
<td>13.0%</td>
<td>0.885 0.783 0.693 0.613 0.543 0.480 0.425 0.376 0.333 0.295 0.261</td>
</tr>
<tr>
<td>14.0%</td>
<td>0.877 0.769 0.675 0.592 0.519 0.456 0.400 0.351 0.308 0.270 0.237</td>
</tr>
<tr>
<td>15.0%</td>
<td>0.870 0.756 0.658 0.572 0.497 0.432 0.376 0.327 0.284 0.247 0.215</td>
</tr>
<tr>
<td>16.0%</td>
<td>0.862 0.743 0.641 0.552 0.476 0.410 0.354 0.305 0.263 0.227 0.195</td>
</tr>
<tr>
<td>17.0%</td>
<td>0.855 0.731 0.624 0.534 0.456 0.390 0.333 0.285 0.243 0.208 0.178</td>
</tr>
<tr>
<td>18.0%</td>
<td>0.847 0.718 0.609 0.516 0.437 0.370 0.314 0.266 0.225 0.191 0.162</td>
</tr>
<tr>
<td>19.0%</td>
<td>0.840 0.706 0.593 0.499 0.419 0.352 0.296 0.249 0.209 0.176 0.148</td>
</tr>
<tr>
<td>20.0%</td>
<td>0.833 0.694 0.579 0.482 0.402 0.335 0.279 0.233 0.194 0.162 0.135</td>
</tr>
<tr>
<td>21.0%</td>
<td>0.826 0.683 0.564 0.467 0.386 0.319 0.263 0.218 0.180 0.149 0.123</td>
</tr>
<tr>
<td>22.0%</td>
<td>0.820 0.672 0.551 0.451 0.370 0.303 0.249 0.204 0.167 0.137 0.112</td>
</tr>
<tr>
<td>23.0%</td>
<td>0.813 0.661 0.537 0.437 0.355 0.289 0.235 0.191 0.155 0.126 0.103</td>
</tr>
<tr>
<td>24.0%</td>
<td>0.806 0.650 0.524 0.423 0.341 0.275 0.222 0.179 0.144 0.116 0.094</td>
</tr>
<tr>
<td>25.0%</td>
<td>0.800 0.640 0.512 0.410 0.328 0.262 0.210 0.168 0.134 0.107 0.086</td>
</tr>
<tr>
<td>26.0%</td>
<td>0.794 0.630 0.500 0.397 0.315 0.250 0.198 0.157 0.125 0.099 0.079</td>
</tr>
<tr>
<td>27.0%</td>
<td>0.787 0.620 0.488 0.384 0.303 0.238 0.188 0.148 0.116 0.092 0.072</td>
</tr>
<tr>
<td>28.0%</td>
<td>0.781 0.610 0.477 0.373 0.291 0.227 0.178 0.139 0.108 0.085 0.066</td>
</tr>
<tr>
<td>29.0%</td>
<td>0.775 0.601 0.466 0.361 0.280 0.217 0.168 0.130 0.101 0.078 0.061</td>
</tr>
<tr>
<td>30.0%</td>
<td>0.769 0.592 0.455 0.350 0.269 0.207 0.159 0.123 0.094 0.073 0.056</td>
</tr>
<tr>
<td>31.0%</td>
<td>0.763 0.583 0.445 0.340 0.259 0.198 0.151 0.115 0.088 0.067 0.051</td>
</tr>
<tr>
<td>32.0%</td>
<td>0.758 0.574 0.435 0.329 0.250 0.189 0.143 0.108 0.082 0.062 0.047</td>
</tr>
<tr>
<td>33.0%</td>
<td>0.752 0.565 0.425 0.320 0.240 0.181 0.136 0.102 0.077 0.058 0.043</td>
</tr>
<tr>
<td>34.0%</td>
<td>0.746 0.557 0.416 0.310 0.231 0.173 0.129 0.096 0.072 0.054 0.040</td>
</tr>
<tr>
<td>35.0%</td>
<td>0.741 0.549 0.406 0.301 0.223 0.165 0.122 0.091 0.067 0.050 0.037</td>
</tr>
<tr>
<td>36.0%</td>
<td>0.735 0.541 0.398 0.292 0.215 0.158 0.116 0.085 0.063 0.046 0.034</td>
</tr>
<tr>
<td>37.0%</td>
<td>0.730 0.533 0.389 0.284 0.207 0.151 0.110 0.081 0.059 0.043 0.031</td>
</tr>
</tbody>
</table>
18.4 CAPITAL (INVESTMENT) APPRAISAL

Capital appraisal (or investment appraisal) is a process used to determine whether long-term investments such as new machinery, replacement of machinery, new plants, new products, and research development projects are worth pursuing.

Many formal methods are used in capital appraisal, and the common ones include:

- The payback period
- The Net present value (NPV)
- Internal rate of return
- Accounting rate of return
- Return on capital employed

This text will only consider the first three.

18.4.1 The payback method

Payback period refers to the period of time required for the return on an investment to "repay" the sum of the original investment. For example, a K200,000 investment which returned K50,000 per year would have a four-year payback period. Payback period intuitively measures how long something takes to "pay for itself." In essence, shorter payback periods are preferable to longer payback periods.

**Example 18**

A business project is being considered which requires K240,000 initial capital outlay. Net revenues over the following 4 years of K80,000.00, K70,000.00, K50,000.00 and K65,000.00 respectively.

Calculate the payback period and state whether the project is worthwhile if the expected payback period is three years or less.

**Solution**

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash flow (K)</th>
<th>Balance (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(240,000.00)</td>
<td>(240,000.00)</td>
</tr>
<tr>
<td>1</td>
<td>80,000.00</td>
<td>(160,000.00)</td>
</tr>
<tr>
<td>2</td>
<td>70,000.00</td>
<td>(90,000.00)</td>
</tr>
<tr>
<td>3</td>
<td>50,000.00</td>
<td>(40,000.00)</td>
</tr>
<tr>
<td>4</td>
<td>65,000.00</td>
<td>25,000.00</td>
</tr>
</tbody>
</table>

At the end of 3 years the project has K40,000.00 to be “repaid”

Hence the project will pay back the initial capital in

\[
\text{years} + \frac{K40,000 \times 12 \text{ months}}{K65,000} = 3 \text{ years and } 7.4 \text{ months}
\]

The project is not viable as its payback period exceeds the desired one (3 years)
Advantages of payback period

Payback period has the following advantages:

i) It is simple to calculate
ii) It is easy to understand
iii) It is practical (It uses actual cash flows) the payback period itself can be set to reflect reality.

Disadvantage of payback period

The technique has one major disadvantage and that is it ignores the time value of money.

18.4.2 The net present value of cash flows

A project is expected to have cash outflows and cash inflows. A net present value (NPV) of a project is the sum of the present values of all cash flows associated with the project.

A project would be considered worthwhile if the net present value is positive.

Example 19

Consider the example 33 and decide whether or not the project is worthwhile using the NPV technique given the cost of capital on the market is 13%.

Solution

Table 18.6 Calculation table – present values

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash flow</th>
<th>df</th>
<th>Present Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(240,000.00)</td>
<td>1.000</td>
<td>(240,000.00)</td>
</tr>
<tr>
<td>1</td>
<td>80,000.00</td>
<td>0.885</td>
<td>70,796.46</td>
</tr>
<tr>
<td>2</td>
<td>70,000.00</td>
<td>0.783</td>
<td>54,820.27</td>
</tr>
<tr>
<td>3</td>
<td>50,000.00</td>
<td>0.693</td>
<td>34,652.51</td>
</tr>
<tr>
<td>4</td>
<td>65,000.00</td>
<td>0.613</td>
<td>39,865.72</td>
</tr>
<tr>
<td></td>
<td>NPV</td>
<td></td>
<td>(39,865.05)</td>
</tr>
</tbody>
</table>

The project’s NPV is negative hence it is not a viable investment.

Example 20

Chikondi is buying a machine costing K120,000 for a small scale business. In order to run it he will spend K85,000 in the first year, K30,000 in the second, K15,000 in the third year and a further K15,000 in the fourth year. Revenues expected from the business are K80,000, K120,000, K100,000 and K65,000 in the 1st, 2nd, 3rd, 4th years respectively. If the cost of capital is at 15% per annum, advise Chikondi as to whether the business is viable.
NPV is positive hence the business is viable.

18.4.3 The Internal Rate of Return (IRR)

The internal rate of return is a discounting rate which would give an NPV of Zero. When calculated the decision on whether to proceed with the project or not is determined by comparing the IRR with the known desired rate of interest. If this rate is higher than the cost of capital or desired rate, the project is considered viable.

Techniques for calculating the IRR

The internal rate of return can be calculated using either algebraic or the graphical techniques

The algebraic method

Example 21
A person invests K500 000 in a project. If the project gives a net inflow of K700 000 for 1 year only, find the IRR. Using the IRR decide if the project is worthwhile given a cost of capital of 12% on the market.

Solution
The IRR is a value r% which gives an NPV of zero

\[
PV_{inflows} - PV_{outflows} = 0 \text{ or } NPV = 0
\]

\[
\frac{K700,000}{\left(1 + \frac{r}{100}\right)^1} - K500,000 = 0
\]

\[
\frac{700,000}{500,000} = 1 + \frac{r}{100}
\]

\[
= 1.4 = 1 + \frac{r}{100}
\]

\[
= r = 40\%
\]

IRR=40%.

Since the cost of capital is the IRR is greater than the cost of capital(12%) then the project is worthwhile.

Therefore IRR =40%. The rate on the market is 12% project worthwhile.
Example 22
A project requires an initial investment of MK500,000 and yields MK300,000 and MK250,000 in the 1st and 2nd years respectively. Calculate the IRR and comment on the project considering the rate of interest on the market is 20%.

Solution:

If the IRR is r% 
Therefore NPV (at r%) = 0

\[-500,000 + \frac{300,000}{1 + \frac{r}{100}} + \frac{250,000}{(1 + \frac{r}{100})^2} = 0\]

\[-10 \left(1 + \frac{r}{100}\right)^2 + 6 \left(1 + \frac{r}{100}\right) + 5 = 0\]

If we let \(i = \frac{r}{100}\)

Then \(-10(1 + i)^2 + 6(1 + i) + 5 = 0\)

\(-10 - 20i - 10i^2 + 6 + 6i + 5 = 0\)

\(10i^2 - 14i - 1 = 0\)

\(i = \frac{-14 \pm \sqrt{14^2 - 4 \times 10 \times -1}}{2 \times 10}\)

\(i = 0.068 \text{ or } -1.468\)

\(\frac{r}{100} = 0.068\)

Hence \(r = 6.8\)

IRR = 6.8% which is less than the rate of interest on the market hence the project is not viable.

The algebraic method works better for projects that do not go beyond 2 years. For projects that go beyond 2 years it is recommended to use the graph or formula to work out the IRR.

The graphical technique

Procedure:

- Choose any two different discount rates different (neither of them should equal the desired discounting rate)
- Draw a pair of axes (NPV against discounting rate). Plot two sets of points (rate and corresponding NPV) on a graph.
• Draw a line through the two points and note the point where the graph cuts the discount rate axis. The value of the discount rate coordinate for this point is the IRR

**Example 23**

Calculate the IRR for the project in example 36 above.

Solution:

Table 18.8 Calculation table – IRR workings

<table>
<thead>
<tr>
<th>Year</th>
<th>Net cash flows</th>
<th>DF(6%)</th>
<th>PV</th>
<th>DF(7%)</th>
<th>PV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(500,000.00)</td>
<td>1</td>
<td>(500,000.00)1</td>
<td>(500,000.00)1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>300,000.00</td>
<td>0.94339623</td>
<td>283018.8679</td>
<td>0.93457944</td>
<td>280,373.83</td>
</tr>
<tr>
<td>2</td>
<td>250,000.00</td>
<td>0.88999644</td>
<td>222499.11</td>
<td>0.87343873</td>
<td>218,359.68</td>
</tr>
<tr>
<td>NPV</td>
<td></td>
<td></td>
<td>5517.977928</td>
<td></td>
<td>-1266.48616</td>
</tr>
</tbody>
</table>

Using formula

The internal rate of return can be calculated as follows

Let

\[ a = \text{first interest rate of interest} \]
\[ A = \text{the NPV corresponds to } a \]
\[ b = \text{the second rate} \]
\[ B = \text{the NPV corresponding to } b \]

Then

\[ IRR = a + \frac{A(b - a)}{(A - B)} \]
Example 24
Calculate the IRR for problem in example 36 using the formula.

Solution
From the solution of example 37, \(a = 6\%; A = 5517.98\)
\(b = 7\%; B = -1266.49\)

\[
IRR = 6 + \frac{(7 - 6) \times 5517.9}{5517.98 - (-1226.49)}
\]

\[= 6.8\%
\]

18.5 ANNUITIES

An annuity is a sequence of fixed amounts at regular intervals.

18.5.1 Classification of annuities

- Based on time of payment/receipt
  - Ordinary annuity—an annuity in which payments are made at the end of the time interval
  - Due annuity—an annuity in which payments are made at the beginning of the time interval
- Based on term of the annuity
  - Certain annuity—dates of beginning and ending of the annuity are fixed
  - Contingent annuity—an annuity whose end date depends on some uncertain event to occur
- Perpetuity—an annuity which goes on indefinitely

Example 25
George is buying a plasma screen worth K460,000 and has agreed to pay a deposit of K100,000 and the balance in 12 equal monthly instalments. Identify the annuity part.

Solution,
Balance after the deposit = K460,000 – 100,000 = 360,000
Instalment: = 360,000/12
= 30,000 per month

The annuity is the sequence of the K30,000s payable every month.

18.5.2 Areas of application

A lot of payments or receipts occur in form of annuities. Good examples are pension payments over time, insurance premium payments, settlement of consumer goods bills inform of instalments for a number of months.

Of interest are the future value to which these sums accumulate and of course given a number of payments, their net present value. Further analyses also deal with the problems of determining how much payments should made be to accumulate to a desired value.

The basic tools in dealing with annuities include the geometric progression, compounding and present
The basic tools in dealing with annuities include the geometric progression, compounding and present values.

### 18.5.3 Future (maturity) value of an annuity

There are basically two methods for calculating the future value of an annuity.

a) Creating an annuity schedule which shows the value of the fund period by period taking into account new payments at each period and all accumulating interests.

b) Considering accrued value of each payment as a term of a geometric progression (GP) and using the formula of sum of terms a GP to find the total maturity value of the payments

**Example 26**

Mr. Yona is investing a fixed amount of K50,000 into a fund every year in advance for 5 years. If the fund earns an interest of 6.5% per annum:

a) Construct a schedule to show the value of the fund at the end of each year.

b) Use an appropriate formula to calculate the value of the fund at the end of the five years.

**Solution**

a) Table 18.9 Calculation table Annuity

<table>
<thead>
<tr>
<th>Year</th>
<th>Payment into the fund</th>
<th>Amount at start of year in the fund</th>
<th>Interest</th>
<th>Amount in the fund (end of year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50,000.00</td>
<td>50,000.00</td>
<td>3,250.00</td>
<td>53,250.00</td>
</tr>
<tr>
<td>2</td>
<td>50,000.00</td>
<td>103,250.00</td>
<td>6,711.25</td>
<td>109,961.25</td>
</tr>
<tr>
<td>3</td>
<td>50,000.00</td>
<td>159,961.25</td>
<td>10,397.48</td>
<td>170,358.73</td>
</tr>
<tr>
<td>4</td>
<td>50,000.00</td>
<td>220,358.73</td>
<td>14,323.32</td>
<td>234,682.05</td>
</tr>
<tr>
<td>5</td>
<td>50,000.00</td>
<td>284,682.05</td>
<td>18,504.33</td>
<td>303,186.38</td>
</tr>
</tbody>
</table>

The amount at the start of the year = The payment for that year plus the cumulated amount at the end of the previous year.

The amount at the end of the year = The amount at the start of the year plus interest earned in the year.

The schedule shows the value of the fund at the end of each year.

b) The future value of the annuity is in form of a sum of terms of geometric progression. Each fixed payment made into the fund will earn interest for the period it is invested.

If A denotes the fixed payments, \( i \) = interest rate:

Future value

\[ A(1 + i)^5 + A(1 + i)^4 + A(1 + i)^3 + A(1 + i)^2 + A(1 + i) \]

Rearranged we have:

\[ A(1 + i)^1 + A(1 + i)^4 + A(1 + i)^3 + A(1 + i)^2 + A(1 + i) \]

290
This is a sum of a geometric progression:

The sum of the first \( n \) terms of a GP is given by:

\[
S_n = \frac{a(r^n - 1)}{r - 1}
\]

where

- \( a \) is the 1st term,
- \( r \) is the common ratio,
- \( n \) is the number of terms.

1. **Finding the 1st term**

   \[
a = A (1 + i)
   \]

   \[
   A = 50,000, \quad \text{therefore} \quad a = 50,000(1.065)
   \]

   \[
   = 53250
   \]

   \[
i = 0.065 \quad r = 1 + i = 1.065
   \]

   The value after 5 years is:

   \[
   S_5 = \frac{53250 \times 1.065^5 - 1}{1.065 - 1}
   \]

   \[
   = 303,186.38
   \]

2. **Present value of an annuity**

Apart from the future value of an annuity, another important concept is the present value of the annuity. The calculation can involve discounting factors or another formula derived from the geometric progression formula.

**Calculation (discounting factors)**

Assuming the same stream of fixed amounts \( A \) set aside for \( n \) years, the present value (actually the net present value) of these amounts can be shown to be:

\[
P = \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \cdots + \frac{A}{(1+i)^n}
\]

Factoring out \( A \) we have:

\[
P = A \left[ \frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^n} \right]
\]

The part in brackets is simply cumulative discounting factor which can be obtained for a calculator or standard tables.

**Example 27**

A man will earn K200,000 in rentals at the beginning of each year for five years. Find the present value of the rentals if the cost of capital is 8%.

**Solution**

\[
P = A \left[ \frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^n} \right]
\]

\[
A = K200,000; \quad i = K0.08
\]

Let the present value be \( P \):

\[
P = 200,000 \left[ \frac{1}{(1.08)^0} + \frac{1}{(1.08)^1} + \frac{1}{(1.08)^2} + \frac{1}{(1.08)^3} + \frac{1}{(1.08)^4} \right]
\]
Not that the rentals are at the start of the year and therefore the first figure has discounting factor = 1
because it is at Present or “now”

Using geometric progression

If viewed a geometric progression point of view:

\[
P = \frac{A}{1 + i} + \frac{A}{(1 + i)^2} + \frac{A}{(1 + i)^3} + \cdots + \frac{A}{(1 + i)^n}
\]

Then

\[
a = \frac{A}{1+i} \quad r = \frac{1}{1+i}
\]

Example 28

A lady is promised a fixed monthly salary of K15,000 for the next 5 years following her injury which incapacitated her. Find the present value of the monthly salaries if the rate of interest on the market is 6% per annum

Solution

This is a case for a geometric progression particularly because of the many terms involved (60) and very few tables can give that many discounting factors. Salaries are paid in arrears.

Monthly rate = \(\frac{6}{12}\)%

= 0.5%

The first term

\[a = \frac{15,000}{(1.005)} = K14925.37\]

The common ratio

\[r = \frac{1}{1.005} = 0.995\]

The number of terms

\[n = 5 \times 12 = 60\]

Present value

\[P = \frac{a(1-r^n)}{1-r} = \frac{K14925.37 \times (1 - 0.995^{60})}{1 - 0.995} = K775,340.26\]

18.5.5 The present value of a perpetuity.

As stated above a perpetuity is an annuity which goes on indefinitely.
For such an annuity the geometric formula with \( n = \infty \) can be used.

**Example 29**
A mineral prospecting company discovers rare earth mineral whose belt runs into the home of Mr. Chimbuzi’s family land. To be able to extract the mineral they must negotiate a payment in order to have the Chimbuzi’s family relocate. They agree on a payment of K100,000 to the family at the beginning of each year for as long as they mine the mineral. Experts estimate that the mineral deposits are in abundance that they cannot estimate the years it will last. Estimate the present value of the payments to the Chimbuzi’s if interest rate is 7%.

**Solution**
It can be assumed that the payment will run for an indefinite period. Therefore \( n = \infty \).

The formula for a sum of GP into perpetuity is

\[
\sum_{i=1}^{\infty} a_i = \frac{a}{1-r}
\]

where \( a = K100,000 \)

\[
r = \frac{1}{1.07} = 0.9346
\]

**Present Value**

\[
= \frac{K100,000}{1-0.9346} = K1,529,051.99
\]

**18.5.6 Amortization and sinking fund.**

Amortisation is the term used to mean reducing or wiping out a debt or a sum of money over time through an annuity. When a debt has been wiped out by amortisation it is said to be amortised.

The amount \( A \) necessary to amortise a debt \( P \) over \( n \) years at an interest rate \( i=r\% \) is given by.

\[
A = \frac{P i}{1-(1+i)^{-n}}
\]

**Example 29**
A company borrows K5, 000, 000 with interest at 5% compounded 6-monthly is amortised by equal semi-annual payments over the next three years. Find the value of each semi-annual payment.

**Solution**
Let the semi-annual payment be \( A \)

\[
A = \frac{P i}{1-(1+i)^{-n}}
\]

\[
= \frac{K5,000,000 \times 0.025}{1-(1.025)^{-6}}
\]

\[
= K907,749.86
\]
A **sinking fund** is really just an annuity set aside to help repay a debt or buy an asset. Typically, one is required to make periodic payments into the fund and normally these payments would earn an interest. The payments would be made in such a way that future value would be sufficient to pay the debt or for the asset.

**Example 30**
Chiko wants to buy a car estimated to cost MK850,000 in 18 months time. The plausible way to save money for it is to set aside a certain amount of money on a monthly basis and invest it at an interest bearing account. The account possible is one at a bank paying an interest rate of 12% per annum compounded monthly.

a) How much should Chiko be setting aside each month,

b) Prepare a sinking fund schedule for the first six months.

**Solution**

a) Interest = 12% pa; therefore monthly rate = 12% ÷ 12 = 1%

Let the amount to be set aside be \( A \)

Assuming the amounts are invested at the start of each month, the future value, \( F \), of the streams of money is

\[
F = A(1.01) + A(1.01)^2 + \cdots + A(1.01)^{18}
\]

This is a geometric progression and the future value is the sum of the \( n \) terms (number of payments or instalments).

\[
F = \frac{a(r^n - 1)}{r - 1}
\]

where \( a = A(1.01) \), \( r = 1.01 \)

but

\[
F = 850,000
\]

\[
k850,000 = \frac{1.01A(1.01^{18} - 1)}{1.01 - 1}
\]

\[
A = \frac{850,000 \times 0.01}{1.01 \times (1.01^{18} - 1)}
\]

\[
A = \frac{850,000 \times (1.01 - 1)}{1.01 \times (1.01^{18} - 1)}
\]

\[
= K42,905.68
\]

(b) The sinking fund schedule is as follows

**Table 18.9 Sinking fund**

294
<table>
<thead>
<tr>
<th></th>
<th>42,905.68</th>
<th>862.40</th>
<th>43,768.09</th>
<th>130,008.51</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>42,905.68</td>
<td>429.06</td>
<td>43,334.74</td>
<td>86,240.42</td>
</tr>
<tr>
<td>2</td>
<td>42,905.68</td>
<td>1,300.09</td>
<td>44,205.77</td>
<td>174,214.28</td>
</tr>
<tr>
<td>3</td>
<td>42,905.68</td>
<td>1,742.14</td>
<td>44,647.83</td>
<td>218,862.11</td>
</tr>
<tr>
<td>4</td>
<td>42,905.68</td>
<td>862.40</td>
<td>43,768.09</td>
<td>130,008.51</td>
</tr>
<tr>
<td>5</td>
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<td>2,188.62</td>
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**CHAPTER SUMMARY**

In this chapter we have looked at the following:

- **Interest: Simple and Compound:**
  - Definition of interest.
  - Calculation of simple interest.
  - Calculation of compound interest.
  - Effective rate of interest.

- **Depreciation:**
  - Definition of depreciation.
  - Depreciating assets using straight line, sum of digits and reducing balance methods of depreciation.

- **Discounting:**
  - Definitions of a present value and discounting factor.
  - Calculation of present values of a future sum.

- **Capital (investment) appraisal using:**
  - Payback period.
  - Net Present Values (NPV).
  - Internal Rate of Return (IRR).

- **Annuities including amortisation and sinking fund**
  - Definition of an annuity.
  - Types of annuities.
  - Calculation of the maturity value of an annuity using a schedule.
  - Calculation of the maturity value of an annuity as a sum of a geometric progression.
  - Debt repayment using amortisation.

Sinking fund as a method for debt repayment and replacement of depreciated assets.
END OF CHAPTER EXERCISES

1. An employee, who received fixed annual increments had a final salary of K90,000 after 10 years. If her total salary was K650,000 over the 10 years what was her initial salary?

2. A firm buys a power press for 32,500 which is expected to last for 20 years and to have a scrap value of K7,500. If depreciation is on a straight line method how much is the depreciation charge per year.

3. Insert 5 numbers between 8 and 26 such that the resulting series is an AP

4. Which term in the GP 2, 4, 8, 16 is 1024

5. A local bank is advertising that you can double your money in eight years if you invest with them. Suppose you have K2000 to invest. What interest (simple) rate is the bank offering

6. Wezi has K50,000 to invest, and two funds that she can use are available: one at Last Discount house offering 14% simple interest and the other at Fist Discount House which pays 6% interest. Wezi only needs to earn K4,500 in interest this year and she really wants to try both banks. How much should she put in each fund

7. A company sets up a sinking fund and invests K10,000.00 each year for 5 years a 9%. Compound interest what will the fund be worth after 5 years?

8. Calculate the sum of the terms in the following sequence: 1, ½, ¼, 1/8, 1/16

9. Mr. Banda’s company is considering an investment costing K5.6 million which would earn K1.6 million cash per annum for five years. The Company expects to make a return of at least 11% per annum.

Assess whether or not the project is viable.

10. At the beginning of each year a company sets aside K2 million out of its profits to form a reserve fund. This is invested at 10% p.a. compound interest. What will be the value of the fund after four years?

11. On 1 January 2003, K0.2 million was invested. It remained invested and on 1 January of each successive year, K0.1 million was added to it. What sum would have accumulated by 31 December 2007 if interest was compounded each year at 10% p.a.?

12. K4 million is borrowed from a building society, repayable over 20 years at 14% p.a. compound interest. How much must be repaid each year?
13. A firm has bought an asset worth K500,000 which has a life of 10 years where its scrap value will be zero. As per its practice the firm will sell the asset at the end of the 5th year and it wishes to select a depreciation technique, which will result in a lowest NBV at the time of selling in order not to overvalue the asset. Calculate the NBV at the end of 5 years using the straight line, sum of digits and reducing balance techniques and suggest the suitable technique, according to the firm's wish.

14. Find the PV of a debt K2500.00 taken out over 4 years (with no intermediate payments) where the borrowing rate is 12% and the worth of money (Discounting rate) is 9.5%. How much is the cost of this debt?

15. Find the present value of 150,000 to be received in 5 years time. Rate of interest is 12% (use the discounting factor method)
CHAPTER 19 CALCULUS

OBJECTIVES

By the end of this chapter students should be able to

i. Differentiate functions up to the second derivative.
ii. Evaluate indefinite and definite integrals.
iii. Find minimum and maximum values of a given function.
iv. Apply calculus on revenue, cost and profit functions with the aim of finding optimum points.

19.0 INTRODUCTION

Calculus is a branch of mathematics which is the study of rate of change of functions.

Using calculus, one can study the rates of change of functions and interpret the results to determine when functions are at their minimum or maximum. For example, we use differentiation to obtain the marginal profit given the cost and revenue functions. The marginal profit function is then used to determine the level of production that will maximize profit.

Calculus is generally divided into two parts: differential and integral calculus.

19.1 DIFFERENTIATION

19.1.1 The process of Differentiation

Given a function y = f(x) the rate of change of f(x) with respect to x is called the derivative of y. The derivative of y = f(x) with respect to x is denoted by

\[ y', f'(x) \text{ or } \frac{dy}{dx} \]

read as y prime, f prime of x and dee y by dee x respectively. The process of finding y' is called differentiation.

19.1.2 Rules for calculating derivatives

There are some rules which enable us to find derivatives quickly and easily.

1. **Power Rule:** If \( y = ax^n \), then \( \frac{dy}{dx} = nax^{n-1} \). Note that the original index “n” multiplies into the function and the new index worked out as “n-1”.

2. **Derivative of a constant:** If \( y = c \), where c is a constant then \( y' = 0 \). That is, the derivative of a constant is zero.

3. **Addition rule:** If \( y = f(x) + g(x) \), then \( y' = f'(x) + g'(x) \). That is, the derivative of a sum is the sum of the derivatives of the individual terms.

Example 1 (General concept of differentiation)

Differentiate the following with respect to x
a) $y = x^3$  
 b) $y = x^6$  
 c) $y = x$

d) $y = 7x$  
 e) $y = 4x^2$  
 f) $y = 3x^{0.5}$

Solution

a) $\frac{dy}{dx} = 3x^2$  
 b) $\frac{dy}{dx} = 6x^5$

c) $\frac{dy}{dx} = 1$  
 d) $\frac{dy}{dx} = 7$

e) $\frac{dy}{dx} = 8x$  
 f) $\frac{dy}{dx} = 1.5x^{-0.5}$

Note that for f) the index is a fraction.

Example 2  Differentiate $y = 12$.

Solution  $\frac{dy}{dx} = 0$.

Example 3  Differentiate with respect to $x$

a) $y = x^{-2}$  
 b) $y = 4/x^3$

Solution

a) $\frac{dy}{dx} = -2x^{-3}$  
 b) $y = 4/x^3 = 4x^{-3}$, so $\frac{dy}{dx} = -12x^{-4} = -12/x^4$

Example 4  Differentiate the following with respect to $x$

a) $y = x^3 - 4x^2 + 13x + 10$  
 b) $y = x^4 - x^{-3} + 2x$

Solution

a) $\frac{dy}{dx} = 3x^2 - 8x + 13$  
 b) $\frac{dy}{dx} = 4x^3 - (-3)x^{-4} + 2 = 4x^3 + 3x^{-4} + 2$

19.1.3 First and second derivatives

When an expression is differentiated for the first time, the result is known as the first derivative. If the result is different from zero, it can be differentiated again to give the second derivative.

The 1st and 2nd derivative are denoted by the following symbols:

<table>
<thead>
<tr>
<th>First derivative</th>
<th>Second derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Usual symbol</td>
<td>$dy/dx$</td>
</tr>
<tr>
<td>Alternative</td>
<td>$y'$</td>
</tr>
</tbody>
</table>

Example 5.  Given $y = 2x^3 + x^2 + 4x - 6$,

$\frac{dy}{dx} = 6x^2 + 2x + 4$ is the first derivative of $y$,

and $\frac{d^2y}{dx^2} = y'' = 12x + 2$ is the second derivative of $y$. 
19.1.4 Interpretation of differentiation
As stated in the introduction, calculus is concerned with the rate of change of a function.

By differentiating a function one actually finds the slope or gradient of that function. Recall that the slope of a function \( y = f(x) \) is the rate at which \( y \) is changing with respect to \( x \).

Example 6
Differentiate the following functions and interpret the results
i) \( y = 4x + 3 \)
ii) \( y = x^2 - x + 4 \)

Solution
i) \( dy/dx = 4 \) This is the slope of the equation (confirm by considering the general linear equation \( y = mx + c \)) where \( c \) is the slope.

ii) \( dy/dx = 2x - 1 \) Because the original equation is not linear, the slope is not a constant but depends on the value of \( x \).

For example, when \( x = 3 \) the slope at that point will be 5 (substitution.)

19.1.5 Maximum and Minimum points
It is possible to investigate whether or not a function has a maximum or minimum point or both.

The process is as follows

Step 1 Find the first derivative \( y' \) or \( f'(x) \).

Step 2 Set the first derivative to zero and solve the resulting equation \( (y' = 0) \).
The roots of \( y' = 0 \) are called critical points.

Step 3 Find the second derivative of the original equation

Step 4 Substitute the values (critical points) found in step 2 into the second derivative.

Step 5 Interpreting the results: If the substitution result in step 4 is negative then the original equation is maximum at the value found in step 2. If the substitution result is positive then the original equation is at its minimum.

Note: We can use the mnemonic NEMA to help us remember that when the second derivative is NEgative at a critical point then the function is MAximum at that point.

Example 7
Given the function \( y = 2x^3 - 3x^2 - 20x + 20 \). Find the maximum and minimum points.

Solution
Following the steps outlined above the process of finding minimum and maximum points is as follows
Step 1  Find the first derivative of the equation
\[ \frac{dy}{dx} = 6x^2 - 6x - 20 \]

Step 2  Set the 1st derivative to zero and solve for x:

\[ 6x^2 - 6x - 20 = 0 \]

Dividing throughout by 3 gives
\[ 3x^2 - x - 10 = 0 \]

This is a quadratic equation and we solve using the formula:

\[ x = \frac{3 \pm \sqrt{9 + 120}}{6} \]

and \[ x = 2.39 \text{ or } x = -1.39 \]

Step 3  Find the second derivative:
\[ \frac{d^2y}{dx^2} = 12x - 6 \]

Step 4  Find the values of the 2nd derivative at the critical points.

At \( x = 2.39 \), \( y' = 12x - 6 = 12(2.39) - 6 = 22.68 \) which is positive.

At \( x = -1.39 \), \( y' = 12x - 6 = 12(-1.39) - 6 = -22.68 \). This is negative.

Step 5  Interpreting and concluding

\[ y = 2x^3 - 3x^2 - 20x + 20 \] is at it’s minimum point when \( x = 2.39 \) and at its maximum when \( x = -1.39 \)

The actual minimum and maximum values of \( y \) are calculated by substituting \( x \) into the original equation.

At \( x = 2.39 \), \( y = 2 \times 2.39^3 - 3 \times 2.39^2 - 20 \times 2.39 + 20 = -17.63 \), so the minimum value is \(-17.63\) and the maximum value is \(36.63\) which occurs when \( x = -1.39 \).

19.1.6 Sketching Graphs
Take a look at fig. 19.1 given below:
This graph is defined for values of x ranging from a to e and has turning points at b, c and d. The turning points take either an \(-\) shape (read as ‘an n shape’) that is at x=b and x=d or a \(\cup\)-shape (‘u shape’) for example at x=c. We can use differentiation to tell where the graph of a function will either of the two shapes.

Generally, a function will have a maximum value at a critical point \(x=a\) and hence an \(-\) shape if the second derivative at that point is negative. Also, if the second derivative of a function at a given point \(x = c\) is positive, then the function will have a minimum value at that point and the graph will assume the \(\cup\)-shape.

Here is a list of steps of how the graph of \(y = f(x)\) can be sketched.

Step 1  Find the x intercept. This is done by solving the equation \(f(x)=0\).
Step 2  Find the y intercept. This is done by finding \(f(0)\).
Step 2  Find the critical points.
Step 2  Use the second derivative to determine which critical points will give rise to a \(\cup\)-shape or \(-\)shape.
Step 3  Plot the points and sketch the graph.

Example 8  Sketch the graph of \(f(x) = 2x^3 + 3x^2 - 36x\).

Solution

Step 1  We solve \(2x^3 + 3x^2 - 36x = 0\) to obtain, \(x = 0\), \(x = 3.56\) and \(x = -5.06\)
Step 2  When \(x = 0\), \(y = 0\). This tells us that the graph will pass through the origin.
Step 2  \(f(x) = 2x^3 + 3x^2 - 36x\) so \(f'(x) = 6x^2 + 6x - 36\) from which the critical points are \(x=-3\) and \(x=2\).
Step 2  \(f''(x) = 12x + 6\) and \(f''(-3) = 12(-3) + 6 = -30 < 0\). So \(x=-3\) is a maximum therefore the graph has a \(-\) shape at \(x=-3\).
At \(x=2\), \(f''(2) = 12(2) + 6 = 30 > 0\), so \(x=2\) is a minimum and so the graph has a \(\cup\)-shape at \(x=2\).
Step 3  Plotting the points gives the graph below.

Figure 19.2  Minimum and maximum points

19.2 INTEGRATION

19.2.1 Introduction
If we are given a function \( y = f(x) \) we know how to find its derivative. In this section we are now going to turn things around. We now want to ask the question, ‘what function did we differentiate in order to get a given function?’ For example, we know that if \( y = x^2 \) then \( y' = 2x \). The question we would like to address is, can we find a function whose derivative \( 2x \)? This question will be answered by carrying out an operation which will undo the effects of differentiation. The reverse of differentiation is called integration.

Let’s go back to the question, can we find a function whose derivative \( 2x \)? It is easy to see that \( y = x^2 \) is one such function. Also \( y = x^2 + 2 \) has its derivative equal to \( 2x \). In fact any function \( y = x^2 + c \) where \( c \) is a constant has a derivative equal to \( 2x \). We can see therefore that we need more information if we are to determine a specific value of \( c \).

Definition 19.2.1 Given a function, \( y = f(x) \), an anti-derivative of \( f(x) \) is any function \( F(x) \) whose derivative is \( f(x) \). That is, \( F'(x) = f(x) \).

If \( F(x) \) is any anti-derivative of \( f(x) \), then the most general anti-derivative of \( f(x) \) is called an indefinite integral and is denoted,

\[
\int f(x)dx = F(x) + c, \text{ c is any constant.}
\]

In this definition the \( \int \) symbol is called the integral symbol, \( f(x) \) is called the integrand, \( x \) is called the integration variable and the “\( c \)” is called the constant of integration. The \( dx \) part specifies the variable of integration.

19.2.2 Rules of Integration
1. \( \int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1 \) and \( c \) is a constant.
2. \( \int kdx = kx + c, k \) and \( c \) are constants.
3. \( \int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx \), the integral of a sum is the sum of the integrals.

**Example 9:** Integrate \( y = x^3 \) with respect to \( x \)

**Solution**

\[ \int x^3 \, dx = \frac{x^4}{4} + c \]

**Example 10:** Integrate \( y = 8x^3 \)

**Solution**

\[ \int (8x^3) \, dx = 8 \frac{x^4}{4} + c = 2x^4 + c \]

**Example 11:** Find the integral of \( y = 5 \) (w.r.t. \( x \))

**Solution**

\[ \int 5 \, dx = 5x + c \]

**Example 12:** Simplify \( \int \frac{2}{x^3} \, dx \)

**Solution**

\[ \int \frac{2}{x^3} \, dx = \int \left( \frac{2}{x^3} \right) \, dx = \frac{2}{x^2} - c + \frac{1}{x^2} + c \]

**Example 13:** Integrate \( y = 6x^4 + x^3 - 2x^4 + 2 \)

**Solution**

\[ \int y \, dx = \int (6x^4 + x^3 - 2x^4 + 2) \, dx = (6x^5)/5 + (x^4)/4 + \left( \frac{2}{3x^3} \right) + 2x + c \]

### 19.2.3 Definite Integrals

**Definition 19.2.3** A definite integral is an integral which has limits within which the integral is to be evaluated. The limits are indicated on the integral sign as follows \( \int_a^b \), where \( a, b \) are the limits.

A definite integral \( \int_a^b f(x) \, dx \), is evaluated as \( \int_a^b f(x) \, dx = F(b) - F(a) \) where \( F(x) \) is an antiderivative of \( f(x) \). That is, \( \int_a^b f(x) \, dx \) is equal to the antiderivative of \( f(x) \) evaluated at the upper limit \( b \) minus the antiderivative of \( f(x) \) evaluated at the lower limit \( a \).

**Example 14** Evaluate \( \int_1^3 (2x^2 + x - 2) \, dx \)

**Solution**

The anti-derivative is \( F(x) = \frac{2x^3}{3} + \frac{x^2}{2} - 2x \).

Now, \( F(3) = \left( \frac{2 \times 3^3}{3} + \frac{3^2}{2} - 2 \times 3 \right) \) and \( F(1) = \left( \frac{2 \times 1^3}{3} + \frac{1^2}{2} - 2 \times 1 \right) \).

\( F(3) - F(1) = (18 + 4.5 - 6) - (0.67 + 0.5 - 2) = 17.33 \).

This means \( \int_1^3 (2x^2 + x - 2) \, dx = 17.33 \).

**Example 15** Evaluate the integral of \( y = 2x^3 + 4 \) between \( x = 0 \) and \( x = 2 \).
Solution Here the problem is \( \int_0^2 (2x^3 + 4)dx \).

Here the anti-derivative is \( F(x) = \frac{2x^4}{4} + 4x \). Now, \( F(2) = \frac{2 \times 2^4}{4} + 4 \times 2 = 16 \), and

\( F(0) = \frac{2 \times 0^4}{4} + 4 \times 0 = 0 \). So \( \int_0^2 (2x^3 + 4)dx = F(2) - F(0) = 16 - 0 = 16 \).

19.3 APPLICATIONS OF SOME CONCEPTS OF CALCULUS

19.3.1 Finding minimum or maximum points in general

Differentiation can be used to find the price that maximizes profit, the dimensions that minimize the cost to construct a box, and the production level that minimizes costs. This will be done in the same manner we approached maximum and minimum values. We will therefore use the second derivative test on a critical value to check whether a critical point maximizes or minimizes a given function.

Example 16 The profit for selling \( x \) hundred units of a product can be approximated by \( P(x) = -x^3 + 45x^2 + 1200x + 80000 \) (up to \( x = 50 \)). What level of sales maximizes the profit?

Solution We begin by finding the derivative: \( P'(x) = -3x^2 + 90x + 1200 \). Now we will set \( P'(x) \) equal to zero and solve for \( x \).

Doing that gives \( -3x^2 + 90x + 1200 = 0 \), from which, after dividing throughout by -3, we obtain \( x^2 - 30x - 400 = (x - 40)(x + 10) = 0 \). Hence, \( x = 40 \) and \( x = -10 \). We disregard the \( x = -10 \) as we cannot sell \( x \) units. What remains now is to check if \( x = 40 \) maximizes the profit.

Now to do this, we need to plug \( x = 40 \) into the second derivative and check if the result is negative. Let’s do that: \( P''(x) = -6x + 90 \) and

\( P''(40) = -6 \times 40 + 90 = -150 \). The conclusion is that \( x = 40 \) maximises since the second derivative is negative (NEMA!).

Example 17 A farmer wishes to make a rectangular chicken run (wire “khola”) using an existing wall as one side. He has 16 meters of wire netting. Find the dimensions of the run which will give the maximum area. What is this area?

Solution Let \( x \) = length of pen opposite existing wall and \( A \) = area

Then the sum of the lengths of the other sides = 16 - \( x \)
One such side has length \( = (16 - x)/2 = 8 - x/2 \)
Figure 19.3 Maximum area of enclosure

existing wall

8-x/2

x

Area A = length * width
= x(8 – x/2) = 8x – x^2/2

dA/dx = 8 - x

Set to zero: 8 – x = 0
x = 8

2nd derivative d^2A/dx^2 = -1 which is negative

Max Area A = 8 * (8 – 8/2)
= 32 m^2

19.3.2 Marginal and Total Functions

Given Total Revenue, Total Cost or Profit functions it is possible to find marginal functions using differentiation. Note that marginal revenue, marginal cost or marginal profit are the rates of change of their total functions. Therefore finding the first derivatives of Total Revenue, Cost and Profit functions gives the marginal revenue, marginal cost or marginal profit functions respectively.

Example 18

The following functions relate to Total cost (TC), Profit (π) and Total revenue (TR) where q is quantity of sales or output.

a) TC = 4q^3 + 7q + 12
b) π = q^2 + 12q + 60
c) TR = 10q – q^2

Required:

i) The marginal and the average functions for each of the total functions.
ii) Evaluate each one of the expressions in i) where q = 3.

Solution

a) i) MC = dTC/dq = 12q^2 + 7
AC = (4q^3 + 7q + 12)/q = 4q^2 + 7 + 12/q

ii) When q = 3
MC = 12 x 3^2 + 7 = 115
AC = 4 x 3^2 + 7 + 12/3 = 47
b) i) \( M = \frac{dJ}{dq} = 2q + 12 \)
\( A = \frac{q^2 + 12q + 60}{q} = q + 12 + \frac{60}{q} \)

ii) When \( q = 3 \)
\( M = 2 \times 3 + 12 = 18 \)
\( A = 3 + 12 + \frac{60}{3} = 35 \)

c) i) \( MR = \frac{dR}{dq} = 10 - 2q \)
\( AC = \frac{10q - q^2}{q} = 10 - q \)

ii) When \( q = 3 \)
\( MR = 10 - 3^2 = 1 \)
\( AR = 10 - 3 = 7 \)

*Note: Average functions are not derived through calculus.*

**Example 19** A firm has the marginal cost = K2.5 with fixed cost of K50,000. The marginal revenue varies with sales and it is given as \( MR = 45 - x \). Find the Total cost and Total revenue functions.

**Solution**
Total cost function is the integral of the marginal cost

\[
TC = \int 2.5 \, dx = 25x + c
\]
When \( x = 0 \)  \( TC = 50,000 \)  Therefore \( c = 50000 \) and \( TC = 2.5x + 50000 \)

\[
TR = \int (45 - x) \, dx = 45x - \frac{x^2}{2} + c
\]
When \( x = 0 \), Revenue is 0 (no sales, no revenue)
Therefore \( c = 0 \)

\( TR = 45x - \frac{x^2}{2} \)

**19.3.3 Revenue and profit maximisation and cost minimisation**

Calculus can be used in revenue and profit maximisation as well as cost minimisation. The only requirement is that these should be in form of non linear functions.

**Example 20** Given the Cost and Revenue functions found in Example 2 above, under 19.4.2. Find:

a) The output that maximises revenue, and profit
b) Find the maximum revenue and maximum profit

**Solution**

a) \( TR = 45x - \frac{x^2}{2} \)
\( d(TR)/dx = 45 - x \)
Set to zero: \( 45 - x = 0 \)  \( x = 45 \)
\( d^2(TR)/dx^2 = -1 \)  Therefore maximum
Revenue is max at $x = 45$

Profit $\pi = TR - TC = (45x - x^2/2) - (2.5x + 50)$

$$= 45x - x^2/2 - 2.5x - 50$$

$$= 42.5x - x^2/2 - 50$$

$$d\pi/dx = 42.5 - x$$

Set equal to zero $42.5 - x = 0$; $x = 42.5$

$$d^2\pi/dx^2 = -1$$

$\pi$ is maximum at $x = 42.5$

**Example 21** Toleza farm are making bricks used as fuel-wood for the community around as part of their social responsibility. The total cost (K‘000) of producing $x$ bricks is defined by the function $C(x) = 20 + 3x$. The farm discovers that the selling price is given by the function: $P = 32 - 2x$. The manager wants you to help him determine:

a) Output which maximises revenue and the maximum revenue

b) Output which maximises profit and the maximum profit

c) Break even output

d) Price at break even

**Solution**

a) Let $R = \text{Revenue}$

$$R = Px = x(32 - 2x) = 32x - 2x^2$$

$$dR/dx = 32 - 4x$$

Set to 0: $32 - 4x = 0$, so $x = 8$.

$$d^2R/dx^2 = -4$$

This is negative, which means Revenue is maximum at $x = 8$

The maximum revenue $R = 32 \times 8 - 2 \times 8^2 = 128$ i.e. K 128, 000

b) Let $\mu = \text{profit}$, and $C = \text{Cost}$

$$\mu = R - C$$ (R is defined in a) above

$$\mu = (32x - 2x^2) - (20 + 3x)$$

$$= 29x - 2x^2 - 20$$

$$d\mu/dx = 29 - 4x$$

Set to 0: $29 - 4x = 0$, $x = 7.25$

$$d^2\mu/dx^2 = -4$$

This is negative, which means $\mu$ is maximum at $x = 7.25$

**Example 22** If additional cost for the bricks above are given by $x^2$. So that the cost function is

$$C(x) = 20 + 3x - x^2$$
Find the point at which costs are minimum.

**Solution**  
\[
\frac{dC}{dx} = 3 - 2x
\]
Setting \(dC/dx\) equal to 0 gives 3 – 2x = 0, and solving for x, we obtain \(x = 1.5\).

\[
d^2J/dx^2 = -4
\]
This is negative, which means J is maximum at \(x = 7.25\).

**Example 23**  
The marginal revenue from the sale of units of a product is \(12 - 0.0004x\). If the revenue from the sale of the first 1000 units is K124,000, find the revenue from the sale of the first 5000 units.

**Solution**  
Here we have \(R'(x) = 12 - 0.0004x\) and that \(R(1000) = 124000\). We are required to find \(R(5000)\). Now, we know that marginal revenue is the derivative of the total revenue so to find the total revenue function, we are going to integrate the marginal revenue. Doing that gives:

\[
\int R'(x)dx = R(x) = \int (12 - 0.0004x)dx = 12x - 0.0002x^2 + c.
\]
Now, since \(R(1000) = 124000\) we have that \(12(1000) - 0.0002(1000)^2 + c = 124000\) so \(c = 112200\). Putting this into the revenue function we obtain \(R(x) = 12x - 0.0002x^2 + 112200\). We are now ready to calculate \(R(5000)\) as we just have to plug \(x = 5000\) into \(R(x)\). \(R(5000) = 12(5000) - 0.0002 \times 5000^2 + 112200 = K167,200\).

**19.4 SUMMARY OF APPLICATIONS OF CALCULUS TO REVENUE COST AND PROFITFUNCTIONS**

<table>
<thead>
<tr>
<th>TABLE 19.1 Summary of revenue, cost and profit functions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FUNCTION</strong></td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

**CHAPTER SUMMARY**

In this chapter we have looked at the following:

- How to differentiate functions using different rules.
- How the first and second derivatives can be used to determine the optimum values of a function.
• How to use critical points and their classifications as local maximum or local minimum to sketch the graph of a given function.
• How to find the integral of a function.
• How the methods of calculus can be used in revenue and profit maximization and cost minimization.

END OF CHAPTER EXERCISES

1. Find the derivatives of the following:
   a) \( y = 2x^2 \) (wrt x)
   b) \( p = 60 - 3q \) (wrt q)
   c) \( y = 4 \) (wrt x)
   d) \( y = x \) (wrt x)
   e) \( y = 2x^3 + 5x^{-2} + 3x + 7 \) (wrt x)

2. Evaluate
   a) \( \int (2x - x^2) \, dx \)
   b) \( \int_2^4 (x^3 - 2x^2 + 5) \, dx \)

3. Suppose the demand curve is: \( p = 18 - 0.5q \)
   a. Find the Total Revenue function (TR).
   b. Find Average Revenue and Marginal Revenue.
   c. Evaluate Marginal Revenue at \( q = 10 \). If MC = 7 should the firm increase production?

4. Suppose a firm has the following profit function.
   \[ \pi = -q^2 + 11q - 24 \]
   Determine the amount of output this firm should produce to maximize its profits.

5. Suppose a firm has the following total revenue and cost functions:
   \[ TR = 1400q - 6q^2 \quad \text{and} \quad TC = 1500 + 80q \]
   Determine the amount of output this firm should produce to maximize its profits.

6. The manufacturing process of a particular item at a company has a total cost function given by \( C = \frac{1}{2}(20x + 6) \) and a revenue function given by \( R = \frac{1}{3}(90x - 10x^2 - 54) \) where \( x \) is the number of items produced (in hundreds), and C and R are both units of K1,000.00
   - Required
     a) Express in its simplest form the total profit \( P \) for the process, where \( P = R - C \)
     b) Plot a graph of \( P \) against \( x \) for values of \( x \) from 0 to 7 at intervals of 1.
     c) Use the graph to determine the values of \( x \) which give:
        i. The maximum profit, giving the value of \( P \).
        ii. The break even points for the process (i.e. where profit \( P = 0 \)).
7. An economist of XYZ limited producing special tucks reckons that for this product weekly revenue and cost functions are

\[ \text{Marginal revenue} = 14 - 2x \]
\[ \text{Marginal cost} = 3x^2 - 15x \]

where \( x \) is the number of tucks in hundreds produced each week. The company has fixed cost of K65.00

(a) Find the equations for Total cost and Total revenue.
(b) Find the total profit function.
(c) For values of \( x = 0 \) to \( x = 8 \), draw a graph of the profit function.
(d) Using your graph write down the estimate of maximum profit.
(e) Use the methods of calculus to find the profit maximizing output and state the maximum profit.

8. Suppose a firm faces the following demand curve

\[ p = 30 - \frac{1}{2}q \]

and has the following total cost curve:

\[ TC = 150 + 10q \]

Required

a) Determine the amount of output this firm should produce to maximize its profits.
b) Use the quantity you found in part (a) (called the “optimal quantity”) to determine the price the good is sold at.
c) Using the optimal quantity, determine the firm’s profits.

9. A manufacturer of a new patented product has found that he can sell 70 units a week direct to the customer if the price is K48. In error, the price was recently advertised at K78 and, as a result, only 40 units were sold in a week. The manufacturer’s fixed costs of production are K1,710 a week and variable costs are K9 per unit. You are required:

a) To show the equation of the demand function linking price \( (P) \) to quantity demanded \( (x) \), assuming it to be a straight line, is \( P = 118 - x \).
b) To find where the manufacturer breaks even.
c) To recommend a unit price which would maximize profit, and to find the quantity demanded and profit generated at the price.
EXAMINATION QUESTIONS EXAMPLES

Examination No.________________________

THE INSTITUTE OF CHARTERED ACCOUNTANTS IN MALAWI

ACCOUNTING TECHNICIAN PROGRAMME

PAPER TC 3: BUSINESS MATHEMATICS

INSTRUCTIONS

Number of questions on paper – 9.

The paper is divided into Sections A and B. ALL questions to be answered in Section A and ANY TWO from Section B.

The maximum number of marks for each answer is indicated against each question.

Mathematical Tables, Formulae Sheets and Graph Paper are provided.

Use of non-programmable calculators is allowed.

Show all your workings in order to gain full marks. Method marks will be awarded throughout.

Begin each answer on a fresh page.

8. **DO NOT OPEN THIS PAPER UNTIL YOU ARE INSTRUCTED BY THE INVIGILATOR.**
SECTION A

ANSWER ALL THE QUESTIONS IN THIS SECTION

1. (a) Add \(6y - 3x - 2y\) to \(6x - y - 2xy\) \(\quad 2\) Marks

(b) Two types of tea costing K650 per kilogram and K720 per kilogram respectively are blended in the ratio 7:3.

Required:

Find the cost of 1 kilogram of the blended tea. \(\quad 5\) Marks

(TOTAL : 7 MARKS)

2. A clerk employed in a certain company, starts with a salary of K120,000 per annum. At the beginning of each year he gets an increment of \(n\) Kwacha per annum so that his salary for the second year is \((120,000 + n)\) Kwacha.

Required:

Find:

(a) the salary that he gets in the third, fourth, fifth and sixth years of his employment. \(\quad 4\) Marks

(b) (i) the total sum of money that he gets for the first five years, in its simplest form. \(\quad 4\) Marks

(ii) the sum if \(n = 60\) \(\quad 4\) Marks

(TOTAL : 8 MARKS)

3 Evaluate:

\(\int_0^2 [1 + 3x(1 + x^2)]dx\) \(\quad 4\) Marks

4. (a) Solve the following inequality: \(x^2 > 25\) \(\quad 8\) Marks

(b) The following data give the quantities and costs of materials for the four divisions of a company for two years.

<table>
<thead>
<tr>
<th>DIVISION</th>
<th>Quantity (tonnes)</th>
<th>Cost (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Year 1</td>
<td>Year 2</td>
</tr>
<tr>
<td>A</td>
<td>175</td>
<td>201</td>
</tr>
<tr>
<td>B</td>
<td>32</td>
<td>46</td>
</tr>
<tr>
<td>C</td>
<td>48</td>
<td>43</td>
</tr>
<tr>
<td>D</td>
<td>65</td>
<td>66</td>
</tr>
</tbody>
</table>

Taking year 1 as the base, calculate:

313
5. (a) A woman goes to the French Cultural Centre every week to watch drama, each time paying K450 for her ticket. She decides that she would pay less if she bought a video cassette recorder for K25,615 and hired one film each week at a cost of K200.

Required:

How many complete weeks will it take to make the new method cheaper than the weekly visits to the French Cultural Centre? 7 Marks

(b) Two shops are offering the same model of TV set for sale which had an original price of K36,000.

Shop A is offering a discount of 8% followed by a special offer of 3% off that discounted price.

Shop B is offering a single discount of 11%.

Required:

What saving would be made by buying the set at the cheaper price? 7 Marks

(TOTAL: 15 MARKS)

6. The following data show the periods (in minutes) that a sample of employees needed to complete a particular task.

| 76 | 59 | 93 | 87 | 38 | 50 | 56 | 113 |
| 102 | 34 | 54 | 85 | 85 | 50 | 45 | 67 |
| 51 | 40 | 82 | 92 | 79 | 38 | 44 | 33 |
| 29 | 107 | 63 | 46 | 68 | 49 | 86 | 34 |
| 61 | 72 | 79 | 45 | 70 | 40 | 99 | 62 |

Required:

(a) Construct a frequency distribution using the class limits 29 – 43, 44 – 58, …

(b) Use your distribution to estimate

(i) the arithmetic mean

(ii) the standard deviation.

(c) Draw a histogram for these data.

(d) Explain whether you would expect the median to be greater than, less than, or about the same as the mean. (NB: You do not need to calculate). 13 Marks

(TOTAL: 13 MARKS)
SECTION B

ANSWER TWO QUESTIONS ONLY FROM THIS SECTION

7. (a) A man wishes to invest K120,000 over a period of 8 years. The “Kulemera Trust” offers a compound interest rate of 8.25%, payable yearly, and “Kusauka Kwatha Investments” offers a rate of 4.5%, payable every 6 months.

Required:
(i) Calculate the accrued amount obtained from investing in each of the two investment companies. 4 Marks
(ii) Which investment company gives the better return, and by how much? 2 Marks
(iii) If the man decides to invest in “Kusauka Kwatha Investments”, how much principal sum will have to be invested to accrue K245,000 over the 8 year period? 5 Marks

TOTAL 11 Marks

(b) The Southern Manufacturing Company completed a feasibility study concerning the development of a new product. The study includes the following estimates.

K’000
Initial cost outlay 1,500
Further outlay at the end of 3 years 800
Residual value after 5 years 500
Net returns at the end of each year for 5 years 550

Required:
(i) Find the net present value. Use a discounting factor of 15%. 9 Marks
(ii) State whether or not the venture is profitable.

(TOTAL : 20 MARKS)

8. (a) State whether the following statements are true or false and give a brief explanation.

(i) A correlation coefficient of –1 means that the appropriate regression line is downward sloping. True
(ii) The mode of a data set is always greater than the median. False
(iii) A quota sample is more accurate than a random sample. True

6 Marks

(b) (i) What causes non-response in a survey?
(ii) Explain how non-response arises in postal/mail surveys conducted randomly.
(iii) You have been asked to conduct a postal survey of people’s holiday plans. Your boss is particularly interested in the popularity of different destinations and types of accommodation. You send out questionnaires during the Christmas break.

Required:
In what ways would you expect non-response to affect your estimates? 11 Marks
(c) You toss a coin twice. What is the probability that you get:
(i) one head and one tail overall?
(ii) A tail on the first throw and a head on the second throw?  

\[ \text{3 Marks} \]

\( \text{TOTAL : 20 MARKS} \)

9. A survey was conducted in 9 areas of Blantyre to investigate the relationship between Divorce rate \( (y) \) and Residential mobility \( (x) \). Divorce rate is the annual number of divorces per 1000 in the population and the Residential mobility is measured by the percentage of the population who have moved house in the last 5 years.

The results of this survey are shown in the table below.

<table>
<thead>
<tr>
<th>Residential Mobility</th>
<th>Divorce Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>3.9</td>
</tr>
<tr>
<td>38</td>
<td>3.4</td>
</tr>
<tr>
<td>46</td>
<td>5.2</td>
</tr>
<tr>
<td>49</td>
<td>4.8</td>
</tr>
<tr>
<td>47</td>
<td>5.6</td>
</tr>
<tr>
<td>43</td>
<td>5.8</td>
</tr>
<tr>
<td>51</td>
<td>6.6</td>
</tr>
<tr>
<td>57</td>
<td>7.6</td>
</tr>
<tr>
<td>55</td>
<td>5.8</td>
</tr>
</tbody>
</table>

Required:

(i) Draw and label the appropriate scatter diagram.  
(ii) Compute the linear regression line of \( y \) on \( x \) for these data.  
(iii) Use this regression equation to estimate the Divorce Rate for an area that has a residential mobility of:

1. 39
2. 60

(iv) Which of these estimates is likely to be more accurate? Give a reason for your answer.

\[ \text{4 Marks} \]

\( \text{TOTAL : 20 MARKS} \)
INSTRUCTIONS

Number of questions on paper – 9.

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Use of non-programmable calculators is allowed.

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Begin each answer on a fresh page.

8. DO NOT OPEN THIS PAPER UNTIL YOU ARE INSTRUCTED BY THE INVIGILATOR.
SECTION A

ANSWER ALL THE QUESTIONS IN THIS SECTION

1. (a) Simplify: \( \frac{1 \frac{13}{27} \times 2 \frac{11}{12} + 3 \frac{1}{9}}{9} \) 2 Marks

(b) K37,000 is to be shared between John, Mary and Jane, whose ages are 10, 12 and 15 years respectively, in the same ratio as their ages.

Required:

How much does each receive? 5 Marks

(TOTAL : 7 MARKS)

2. (a) Solve the following equation for \( x \):

\[ \frac{2x + 3}{2} - \frac{2(x - 4)}{3} = 7 \frac{5}{6} \]

5 Marks

(b) A salesman has to decide between two jobs. The first job offers a salary of K15,000 per week plus a commission of 2% of sales and the second offers a salary of K7,500 per week plus 5% of sales.

Required:

If he estimates that he is likely to sell K12,500,000 worth of goods in a year, which job will give him the greater return for the year and by how much? 8 Marks

(TOTAL : 13 MARKS)

3. (a) Find the equation of the line passing through (4, 12) and (8, -2). 3 Marks

(b) 20 shirts and 6 ties cost a shopkeeper K5,300 and 18 shirts and 20 ties cost him K5,500.

Required:

Find the cost of one shirt and of one tie. 8 Marks

(TOTAL : 11 MARKS)

4 A mower is marked for sale at K52,000. A man buys this mower after obtaining a K12,000 trade-in on his old mower. He agrees to pay off the mower in 18 equal monthly payments. The shop calculates that he will pay K9,000 interest.

Required:

Calculate the following:

(a) The total amount to be paid. 3 Marks
(b) The amount of each repayment. 2 Marks
(c) The rate of interest being charged. 3 Marks

(TOTAL : 8 MARKS)
5. Kesale Estate Agents released the following daily sales figures for the past 14 weeks.

<table>
<thead>
<tr>
<th>Houses sold</th>
<th>Number of days on which sales were made</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>6 or more</td>
<td>2</td>
</tr>
</tbody>
</table>

Required:
(a) How many sales were made over the 14 week period? 2 Marks
(b) Find the mode, median and mean number of houses sold. 5 Marks
(c) If the average selling price was K1,160,000 and the agents’ commission was 10%, how much commission was earned over the period? 3 Marks

(TOTAL : 10 MARKS)

6. A company is faced with the following marginal cost and marginal revenue functions:

\[ MC = 16 - 2Q \]
\[ MR = 40 - 16Q \]

Where Q is the production.

It is also known that fixed costs are 8 when production is zero.

Required:

Find:

(a) The total cost function. 4 Marks
(b) The total revenue function. 4 Marks
(c) The output to give the maximum sales revenue. 3 Marks

(TOTAL : 11 MARKS)
SECTION B

ANSWER TWO QUESTIONS ONLY FROM THIS SECTION

7. Linda Furniture Ltd released the following table of goods against stock-on-hand for the financial year 2005/06.

<table>
<thead>
<tr>
<th>Month</th>
<th>Average stock on hand</th>
<th>Average storage costs (MK'000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>July</td>
<td>34</td>
<td>73</td>
</tr>
<tr>
<td>August</td>
<td>12</td>
<td>28</td>
</tr>
<tr>
<td>September</td>
<td>17</td>
<td>40</td>
</tr>
<tr>
<td>October</td>
<td>7</td>
<td>20</td>
</tr>
<tr>
<td>November</td>
<td>8</td>
<td>23</td>
</tr>
<tr>
<td>December</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>January</td>
<td>11</td>
<td>27</td>
</tr>
<tr>
<td>February</td>
<td>17</td>
<td>42</td>
</tr>
<tr>
<td>March</td>
<td>14</td>
<td>30</td>
</tr>
<tr>
<td>April</td>
<td>26</td>
<td>54</td>
</tr>
<tr>
<td>May</td>
<td>20</td>
<td>48</td>
</tr>
<tr>
<td>June</td>
<td>28</td>
<td>60</td>
</tr>
</tbody>
</table>

The following statistics can be used: \[ \sum xy = 9333 : \sum x^2 = 4177; \sum y^2 = 20999 \]

Required:

(a) Draw a scatter diagram for the data.  

(b) (i) Calculate the Product Moment Correlation Coefficient (r) for the data.  
       (ii) Interpret the value of r.  

(c) (i) Find the least squares regression line of storage costs on stock-on-hand.  
       (ii) Plot your regression line on the scatter diagram.  
       (iii) Predict the storage cost if the average stock-on-hand was 30 units.  

(TOTAL : 20 MARKS)

8. (a) A company owns a fleet of 100 cars, each of which has either manual or automatic transmission and either 2 or 4 doors. There are 25 cars which are 2 door models and of these 10 have automatic transmission. There are 35 cars with manual transmission.

Required:

If a car is picked at random from the fleet, calculate the probability that it is:

   (i) Automatic;  
   (ii) 2 – door;  
   (iii) Automatic or 4 – door;
(iv) Automatic and 4 – door;
(v) Automatic, given it is 2 – door;
(vi) 2 – door, given it is automatic;

(Give your answers to 2 decimal places).

7 Marks

(b) A dealer buys a computer, costing him K90,000 and sells it to a company, making a profit of 45%.

Required:

(i) At what price does the dealer sell the computer?  

2 Marks

(ii) The company estimates that the computer will have a useful life of 4 years, depreciating at a rate of 25% per annum.

Required:

What is the residual value of the computer after 4 years? (Give your answer to the nearest Kwacha).  

3 Marks

(c) (i) Chiyembekezo has taken out a mortgage of K600,000 to be paid over 25 years. Interest is to be charged at 12% p.a.

Required:

Calculate the monthly repayment.  

3 Marks

(ii) After nine years, the interest rate changes to 10% p.a. What is the new monthly repayment?  

5 Marks

(TOTAL : 20 MARKS)

10. The table below gives the life of a particular brand of a car tyre (in kilometers travelled) obtained from a road service survey.

<table>
<thead>
<tr>
<th>Kilometers travelled</th>
<th>Number of tyres</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 3,999</td>
<td>1</td>
</tr>
<tr>
<td>4,000 - 7,999</td>
<td>3</td>
</tr>
<tr>
<td>8,000 - 11,999</td>
<td>8</td>
</tr>
<tr>
<td>12,000 - 15,999</td>
<td>14</td>
</tr>
<tr>
<td>16,000 - 19,999</td>
<td>20</td>
</tr>
<tr>
<td>20,000 - 23,999</td>
<td>15</td>
</tr>
<tr>
<td>24,000 - 27,999</td>
<td>10</td>
</tr>
<tr>
<td>28,000 - 31,999</td>
<td>6</td>
</tr>
<tr>
<td>32,000 - 35,999</td>
<td>2</td>
</tr>
<tr>
<td>36,000 - 39,999</td>
<td>1</td>
</tr>
</tbody>
</table>

Required:
(a) Draw a cumulative frequency distribution curve to represent these data.  
8 Marks

(b) From the curve, determine the median and the upper and lower quartiles for these data.  
3 Marks

(c) Determine the interquartile range for these data and explain what it shows. State why this measure is preferred to other measures of variation.  
5 Marks

(d) What percentage of the tyres have a life

(i) above 25,000 km;  
(ii) below 15,000 km?  
4 Marks

(TOTAL : 20 MARKS)
INSTRUCTIONS

1. Number of questions on paper - 9.

2. The paper is divided into Sections A and B. ALL questions to be answered in Section A and ANY TWO from Section B.

3. The maximum number of marks for each answer is indicated against each question.

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7. Begin each answer on a fresh page.

8. DO NOT OPEN THIS PAPER UNTIL YOU ARE INSTRUCTED BY THE INVIGILATOR.
SECTION A

ANSWER ALL QUESTIONS IN THIS SECTION

1. (a) Simplify:

\[
\frac{3x^2 + 2}{3x + 2} - 1
\]

3 Marks

(b) The daily wage rates of three women, Mary, Maria and Margret, are in the ratios 24:16:35.

Required:

Calculate:

(i) The total value of the wages earned, assuming that Mary earned K1,200; 4 Marks

(ii) Maria’s and Margaret’s wages. 4 Marks

(TOTAL: 11 MARKS)

2. (a) Solve the following simultaneous equations:

\[
x + y = 1; \\
x + 2y = 5.
\]

5 Marks

(b) A company has bought an asset which has a life span of four years. At the end of the four years, a replacement asset will cost K120,000 and the company has decided to provide for this future commitment by setting up a sinking fund into which equal annual investments will be made, starting at year 1 (one year from now). The fund will earn interest at 12%.

Required:

Calculate the annual repayments. 3 Marks

(TOTAL: 8 MARKS)

3. (a) Given that matrix

\[
A = \begin{bmatrix} 8 & 9 \\ 12 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 13 & 4 \\ 2 & 6 \end{bmatrix}
\]

Required:

Find:

(i) matrix C such that C = A.B; 3 Marks

(ii) the inverse of A. 5 Marks

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(b) A firm keeps details of component parts used in the make up of each product (Matrix K) and products made on each day of the week (Matrix M) as follows:

<table>
<thead>
<tr>
<th>Product</th>
<th>Parts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>Mon</td>
<td>0</td>
</tr>
<tr>
<td>Tue</td>
<td>2</td>
</tr>
<tr>
<td>Wed</td>
<td>3</td>
</tr>
<tr>
<td>Thu</td>
<td>1</td>
</tr>
<tr>
<td>Fri</td>
<td>1</td>
</tr>
</tbody>
</table>

Required:

Using matrix multiplication, find a matrix that describes the number of component parts used on each day of the week.  

(TOTAL: 13 MARKS)

4. (a) A machine costing K256,500 depreciates to a scrap value of K5,000 in ten years.

Required:

Calculate:

(i) The annual percentage rate of depreciation using the reducing balance method of depreciation.  

(ii) The book value at the end of the sixth year using the reducing balance method of depreciation.

(b) It is estimated that a mine will yield an annual net return (i.e after all operating costs) of K500,000 for the next 15 years. At the end of this time the property will be valueless.

Required:

Calculate the purchase price of the mine if the purchase price is equivalent to the present value of a K500,000 annuity over 15 years at 12% discount rate.

(TOTAL: 11 MARKS)

5. Sampling methods are widely used for the collection of statistical data in industry and business.

Required:

Illustrating your answers with practical examples, explain the following terms: